ON THE FIRST ORDER SECULAR PERTURBATIONS OF AN ARTIFICIAL SATELLITE IN THE GRAVITATIONAL FIELD OF THE OBLATE EARTH

by

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Some interesting theoretical problems have arisen concerning the revolution of artificial satellites around the Earth. This kinematic phenomenon could only be described accurately by Kepler's well-known laws if no force other than the gravitational attraction of an Earth of spherically symmetrical mass-distribution acted upon the artificial satellite. It is obvious, however, that in the case of a celestial body moving in the proximity of the Earth's surface neither air resistance, nor the correction of the gravitational field due to the oblateness of the Earth may be neglected. At greater distances, on the other hand, the attraction of the Moon and the Sun must be taken into account. While the theory of perturbations due to air resistance is easily accessible in past astronomical literature (*e. g.* [1]), the problem of the perturbations caused by oblateness has only recently arisen in the general form needed in our case. In the present paper, after a comparison of the gravitational forces to be taken into account, the first order secular perturbations of an artificial satellite due to the oblateness of the Earth shall be determined for an arbitrary inclination and excentricity of its orbit.

1. A COMPARISON OF THE GRAVITATIONAL FORCES ACTING UPON AN ARTIFICIAL SATELLITE

If the oblateness of the Earth were negligible, or if we were to examine its attraction at a great distance from it, the gravitational field of a point-mass with a mass equal to that of the Earth and located at the centre of the Earth could be substituted for its actual gravitational field. In our case, however, the attraction of the Earth can be described with sufficient accuracy by the force-function

$$U(r, \beta) = \frac{\mu}{r} \left\{ 1 + \frac{\nu}{r^2} (1 - 3 \sin^2 \beta) \right\} \tag{1}$$

where

r is the distance of a point in space from the centre of the Earth,

β is the angle between the radius vector and the plane of the equator,

$$\mu = f\, m_E \quad \text{and} \quad \nu = \frac{C - A}{2\, m_E}$$

where

f is the gravitational constant,

m_E the mass of the Earth,

C and A its moments of inertia about the axis of rotation and an equatorial axis, respectively (*e. g.* [2], apart from notation).

The second term of the force-function (1) yields the disturbing function R due to the oblateness of the Earth, the disturbing force acting upon a unit mass being its gradient $\mathbf{F}$.
Since

$$R = \frac{\mu \nu}{r^3}(1 - 3 \sin^2 \beta)$$

depends only on the geocentric distance of the artificial satellite and on its latitude the vector $\mathbf{F}$ always lies in a meridian plane. Namely

$$\mathbf{F} = \operatorname{grad} R = \frac{\partial R}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial R}{\partial \beta}\mathbf{e}_\beta$$

where $\mathbf{e}_r$ and $\mathbf{e}_\beta$ are unit vectors normal to the sphere $r =$ const. and the cone $\beta =$ const. respectively in the corresponding point of space. Thus, on account of

$$\frac{\partial R}{\partial r} = -\frac{3 \mu \nu}{r^4}(1 - 3 \sin^2 \beta)$$

and

$$\frac{1}{r}\frac{\partial R}{\partial \beta} = -\frac{6 \mu \nu}{r^4} \sin \beta \cos \beta$$

we have

$$|\mathbf{F}| = \frac{3 \mu \nu}{r^4}\sqrt{1 - 2 \sin^2 \beta + 5 \sin^4 \beta}$$

Now, $$1 - 2 \sin^2 \beta + 5 \sin^4 \beta = \frac{4}{5} + \left(\frac{1}{\sqrt{5}} - \sqrt{5} \sin^2 \beta\right)^2$$

and therefore

$$\min |\mathbf{F}| = \frac{6 \mu \nu}{\sqrt{5} r^4} \tag{2}$$

on the sphere $r =$ const.

In order to take into account the disturbing force of the Sun and the Moon, the position vectors $\mathbf{r}_M$, $\mathbf{r}_S$ of these celestial bodies shall also be related to the centre of the Earth. In this case, the equations of motion can be obtained in the conventional way in the form

$$\frac{d^2 \mathbf{r}}{dt^2} + \frac{\mu \mathbf{r}}{r^3} = \mathbf{F} + \mathbf{F}_M + \mathbf{F}_S \tag{3}$$

where

$$\mathbf{F}_M = \mu_M \left\{\frac{\mathbf{r}_M - \mathbf{r}}{|\mathbf{r}_M - \mathbf{r}|^3} - \frac{\mathbf{r}_M}{r_M{}^3}\right\}, \quad \mathbf{F} = \mu_s \left\{\frac{\mathbf{r}_s - \mathbf{r}}{|\mathbf{r}_s - \mathbf{r}|^3} - \frac{\mathbf{r}_s}{r_s{}^3}\right\} \qquad (3_M, 3_S)$$

and $\mu_M = fm_M$, $\mu_s = fm_s$. To estimate the Moon's disturbing force, let us introduce the unit vector $\mathbf{e}_M = \mathbf{r}_M / r_M$ directed at the Moon, and the angle ψ_M between the vectors $\mathbf{r}$ and $\mathbf{r}_M$. Then

$$\frac{1}{|\mathbf{r}_M - \mathbf{r}|^3} = \frac{1}{r_M^3 \, |\mathbf{e}_M - \mathbf{r}/r_M|^3} = \frac{1}{r_M^3}\left[1 - 2\left(\frac{r}{r_M}\right)\cos\psi_M + \left(\frac{r}{r_M}\right)^2\right]^{-\frac{3}{2}} =$$

$$= \frac{1}{r_M^3}\left[1 + 3\left(\frac{r}{r_M}\right)\cos\psi_M \pm \ldots\right]. \qquad (4)$$

In our case $r/r_M \approx 1/60$, so the unwritten terms may be safely neglected. Now, on account of (3_M) and (4) it can be stated with sufficiant accuracy that

$$\mathbf{F}_M = \frac{\mu_M r}{r_M^3}\{3\cos\psi_M \, \mathbf{e}_M - \mathbf{e}_r\}, \qquad |\mathbf{F}_M| \doteq \frac{\mu_M r}{r_M^3}\sqrt{1 + 3\cos^2\psi_M}\,.$$

consequently

$$\max|\mathbf{F}_M| = \frac{2\,\mu_M r}{r_M^3}. \qquad (5_M)$$

Similarly,

$$\max|\mathbf{F}_S| = \frac{2\,\mu_S r}{r_S^3} \qquad (5_S)$$

Let us compare the values (2), (5_M), $5_S)$ obtained for the disturbing forces with the attractive force of an Earth supposed to be spherical. The value

$$\frac{C - A}{m_E \, \mathrm{a}^2} = 0{,}001106 \pm 0{,}00001$$

can be deduced from the perturbations of the Moon's motion [3], where a is the equatorial radius of the Earth; hence $\nu = 5{,}53 \cdot 10^{-4} \times \mathrm{a}^2$. According to Hayford, a $= 6378{,}388$ km. Furthermore, $m_E/m_M = 81{,}53$ $r_M/\mathrm{a} = 60{,}31$ and from the data of the "Berliner Astronomisches Jahrbuch" $m_s/m_E = = 331930$, $r_s/\mathrm{a} = 23439$. Using these values

$$\min|\mathbf{F}| : \frac{\mu}{r^2} = \frac{6\,\nu}{\sqrt{5}\,\mathrm{a}^2}\left(\frac{\mathrm{a}}{r}\right)^2 = 1{,}484 \cdot 10^{-3} \times \left(\frac{r}{\mathrm{a}}\right)^{-2}$$

$$\max|\mathbf{F}_M| : \frac{\mu}{r^2} = 2\,\frac{m_M}{m_E}\left(\frac{\mathrm{a}}{r_M}\right)^3\left(\frac{r}{\mathrm{a}}\right)^3 = 1{,}118 \cdot 10^{-7} \times \left(\frac{r}{\mathrm{a}}\right)^3$$

$$\max|\mathbf{F}_S| : \frac{\mu}{r^2} = 2\,\frac{m_S}{m_E}\left(\frac{\mathrm{a}}{r_S}\right)^3\left(\frac{r}{\mathrm{a}}\right)^3 = 5{,}155 \cdot 10^{-8} \times \left(\frac{r}{\mathrm{a}}\right)^3$$

We note that the relative disturbing force caused by the oblateness of the Earth decreases with the square of geocentric distance, while the relative disturbing forces of the Sun and the Moon increase with its cube. In the immediate vicinity of the Earth's surface $|\mathbf{F}| : |\mathbf{F}_M| > 13000$, but even at a height of 10000 kilometres above we have $|\mathbf{F}| : |\mathbf{F}_M| > 100$. In this range $\mathbf{F}_M$ and

$\mathbf{F}_S$ (the latter being about half the amount of the former) may be neglected beside $\mathbf{F}$, if the motion of the artificial satellite is to be followed for a short time only (*e. g.* one month). For the sake of completeness we note that the unwritten third term of the force-function (1) obtained by series expansion is about 400 times smaller than the second term.

2. THE DETERMINATION OF SECULAR PERTURBATIONS DUE TO OBLATENESS

Only two integrals of the equations of motion

$$\frac{d^2\mathbf{r}}{dt^2} = \operatorname{grad}\left(\frac{\mu}{r} + R\right)$$

of the artificial satellite can be given. Firstly, since R does not explicity depend on time, the energy integral

$$\frac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2 - \left(\frac{\mu}{r} + R\right) = H \tag{6}$$

is valid. Secondly, the component of the angular momentum parallel to the Earth's axis, or if you prefer, the double area velocity of the motion projected onto the plane of the equator, is a constant:

$$\mathbf{e}_N\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = G \tag{7}$$

where $\mathbf{e}_N$ is a unit vector directed at the North Pole. Though these integrals provide valuable informations concerning the nature of the motion, they are far from sufficient to describe it completely. Therefore one of the approximative methods of the calculus of perturbations based on the comparative smallness of deviation from Keplerian motion must be resorted to. Those differential equations seem to be most suitable for our purpose that relate the change of osculating orbital elements directly to the radial component S, the transversal component T, and the orthogonal component W [4, 5] of the disturbing force. In particular, we shall need the following differential equations:

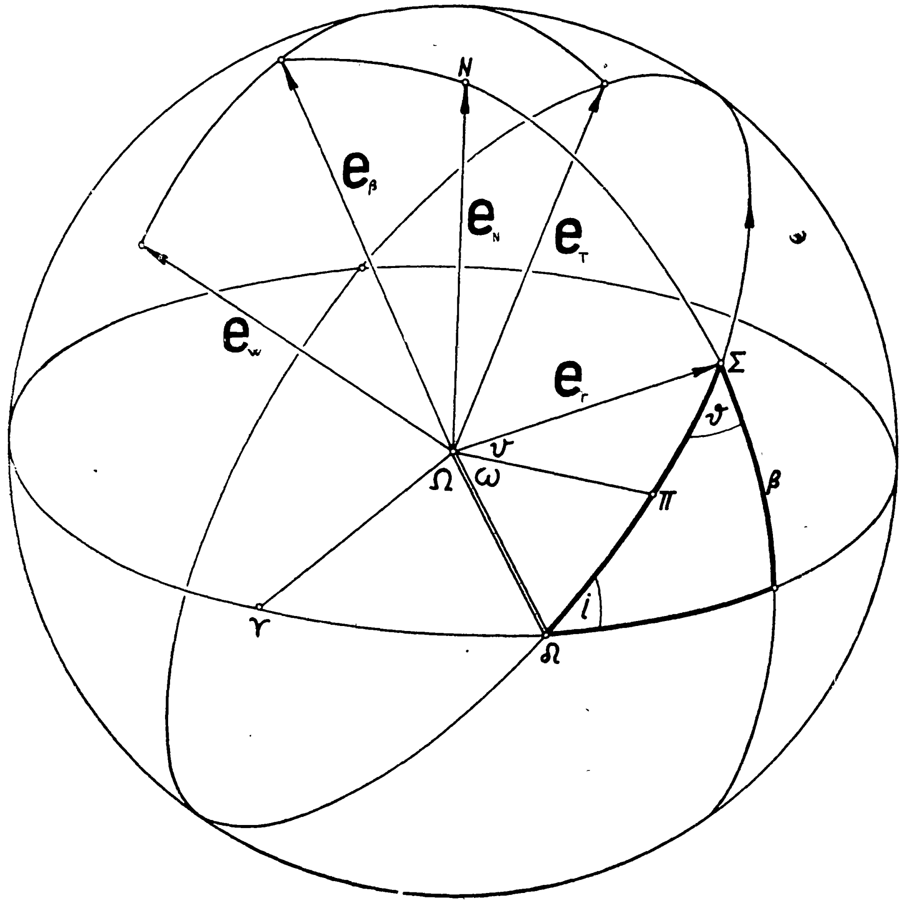
$$\frac{di}{dt} = \frac{Wr}{\sqrt{\mu p}}\cos(\omega + v) \qquad \sin i\,\frac{d\Omega}{dt} = \frac{Wr}{\sqrt{\mu p}}\sin(\omega + v)$$

$$\frac{d\omega}{dt} + \cos i\,\frac{d\Omega}{dt} = \frac{1}{e}\sqrt{\frac{p}{\mu}}\left\{T\left(1 + \frac{r}{p}\right)\sin v - S\cos v\right\} \tag{8}$$

$$\frac{da}{dt} = \frac{2a^2}{\sqrt{\mu p}}\left\{T\,\frac{p}{r} + eS\sin v\right\}$$

where Ω is the length of the ascending node of the momentary orbit plane measured from the first point of Aries, and ω is the angular distance of the perigee from the ascending node. The meaning of the orbital elements i, a, e, $p = a(1 - e^2)$ is evident. The true anomaly of the artificial satellite, measured from the perigee, is to be designated v; let $u = \omega + v$.

In order to apply the differential equations (8) the components S, T, W of the vector $\mathbf{F}$ must be computed. The radial unit vector $\mathbf{e}_\gamma$ and the meridian tangent unit vector $\mathbf{e}_\beta$ have been introduced already. If furthermore $\mathbf{e}_W$ shall be the unit vector normal to the orbit plane, and $\mathbf{e}_T = \mathbf{e}_W \times \mathbf{e}_r$, the components in question will respectively be



$$S = \mathbf{F}\mathbf{e}_r = \frac{\partial R}{\partial r}, \quad T = \mathbf{F}\mathbf{e}_T = \frac{1}{r}\frac{\partial R}{\partial \beta}\mathbf{e}_\beta\,\mathbf{e}_T\,, \quad W = \mathbf{F}\mathbf{e}_W = \frac{1}{r}\frac{\partial R}{\partial \beta}\mathbf{e}_\beta\,\mathbf{e}_W\,.$$

In the diagram these four unit vectors are represented as being located in the origin of the coordinate system. Denoting the inclination angle of the meridian circle and the orbit plane by ϑ, it is plain that $\mathbf{e}_\beta\,\mathbf{e}_T = \cos\vartheta$ and $\mathbf{e}_\beta\,\mathbf{e}_W = \sin\vartheta$.

Since the equation of the osculating ellipse is

$$r = \frac{p}{1 + e\cos v}$$

the components S, T, W are expressed on account of the foregoing by means of the angles v, β, ϑ, which all change rapidly during the motion. But for the rectangular spherical triangle denoted by thick lines we have the relations

$$\sin\beta = \sin i \;\sin u$$
$$\cos\beta \;\cos\vartheta = \sin i \;\cos u$$
$$\cos\beta \;\sin\vartheta = \cos i$$

so in the end

$$S = -\frac{3\,\mu\,\nu}{r^4}(1 - 3\sin^2\beta) = -\frac{3\,\mu\,\nu}{r^4}(1 - 3\sin^2 i\;\sin^2 u)$$

$$T = -\frac{6\,\mu\,\nu}{r^4}\sin\beta\cos\beta\cos\vartheta = -\frac{6\,\mu\,\nu}{r^4}\sin^2 i\;\sin u\;\cos u \tag{9}$$

$$W = -\frac{6\,\mu\,\nu}{r^4}\sin\beta\cos\beta\sin\vartheta = -\frac{6\,\mu\,\nu}{r^4}\sin i\;\cos i\;\sin u\,.$$

Since now it is only the angle u, or if you prefer v that changes rapidly during the motion of the artificial satellite, while the orbital elements are constant in the 0-th approximation, the expressions (9) can be substituted into the differential equations (8), yielding easily manageable perturbational equations.

In order to integrate, an other independent variable than time must of course be chosen, in our case this variable is evidently the true anomaly. It must be mentioned, that Kepler's 2-nd law does not hold exactly now, rather it is related to time by the expression [6] :

$$\frac{d\,v}{d\,t} = \frac{\sqrt{\mu\,p}}{r^2} - \left(\frac{d\,\omega}{d\,t} + \cos i\,\frac{d\,\Omega}{d\,t}\right).$$

But in order to determine first-order perturbations, $r^2 dv/\sqrt{\mu\,p}$ may be substituted for dt in (8). Thus the final form of our differential equations will be

$$\frac{d\,i}{d\,v} = -\frac{6\,\nu}{p^2}\sin i\;\cos i\,[\sin u\;\cos u\,(1 + e\cos v)]$$

$$\frac{d\,\Omega}{d\,v} = -\frac{6\,\nu}{p^2}\cos i\,[\sin^2 u\,(1 + e\cos v)]$$

$$\frac{d\,\omega}{d\,v} + \cos i\,\frac{d\,\Omega}{d\,v} = -\frac{3\,\nu}{e\,p^2}\{2\sin^2 i\,[\sin u\cos u\sin v\,(2 + 3\,e\cos v + e^2\cos^2 v)] - $$
$$- (1 - 3\sin^2 i\;\sin^2 u)\cos v\,(1 + 2\,e\cos v + e^2\cos^2 v)\}$$

$$\frac{d\,a}{d\,v} = -\frac{6\,\nu\,a^2}{p^3}\{2\sin^2 i[\sin u\;\cos u\,(1 + 3\,e\cos v + 3\,e^2\cos^2 v + e^3\cos^3 v)] +$$
$$+ e\,(1 - 3\sin^2 i\;\sin^2 u)\sin v\,(1 + 2\,e\cos v + e^2\cos^2 v)\}\,. \tag{10}$$

Now the first-order perturbations may be determined by regarding the orbital elements at the right side of these differential equations as constant (since the comparatively slight disturbing force causes them to change but slowly) and by integrating according to v. Since we are dealing with trigonometric polynomials, this integration presents no basic difficulty. After these integrations we get the first order perturbations. To follow the motion of artificial satellites a comprehensive knowledge of the secular variations of the orbital elements is most important. In order to compute secular perturbations arising with the present approximation we need only the values of the definite integrals from 0 to $2\,\pi$ of true anomaly. And this can be obtained with the aid of a very short calculation.

To begin with, we shall note that on the right-hand side of the differential equations (10) the integrals from 0 to 2π of all those terms equals 0 in which the sum of the exponents of $\sin u$, $\cos u$, $\sin v$, and $\cos v$ is an odd number. So the only integrals to be taken into account are :

$$\int_0^{2\pi} \sin u \cos u \, dv = 0\,, \qquad \int_0^{2\pi} \sin^2 u \, dv = \pi$$

$$\int_0^{2\pi} \sin u \cos u \sin v \cos v \, dv + \int_0^{2\pi} \sin^2 u \cos^2 v \, dv = \int_0^{2\pi} \sin u \cos v \sin (u + v) \, dv =$$

$$= \frac{1}{2} \int_0^{2\pi} \sin^2 (2\,v + \omega) \, dv + \frac{1}{2} \sin \omega \int_0^{2\pi} \sin (2\,v + \omega) \, dv = \frac{\pi}{2}$$

$$\int_0^{2\pi} \sin u \cos u \cos^2 v \, dv - \int_0^{2\pi} \sin^2 u \sin v \cos v \, dv = \int_0^{2\pi} \sin u \cos v \cos (u + v) \, dv =$$

$$= \frac{1}{2} \int_0^{2\pi} \sin (2\,v + \omega) \cos (2\,v + \omega) \, dv + \frac{1}{2} \sin \omega \int_0^{2\pi} \cos (2\,v + \omega) \, dv = 0\,,$$

$$\int_0^{2\pi} \cos^2 v \, dv = \pi \qquad \int_0^{2\pi} \sin v \cos v \, dv = 0\,.$$

So the secular perturbations of the orbital elements i, Ω, ω, and a during one complete revolution are

$$\delta\, i = 0\,, \quad \delta\Omega = -\frac{6\,\pi\nu}{p^2} \cos i\,, \quad \delta\omega + \cos i\, \delta\Omega = \frac{6\,\pi\nu}{p^2} \left(1 - \frac{3}{2} \sin^2 i\right)$$

$$\delta\omega = \frac{12\,\pi\nu}{p^2} \left(1 - \frac{5}{4} \sin^2 i\right), \qquad \delta\, a = 0\,. \tag{11}$$

Or, expressed in degrees and employing the constant $\nu = 5{,}53 \cdot 10^{-4} \times \mathrm{a}^2$:

$$\delta\Omega = -\,0°,597 \left[\frac{\mathrm{a}}{a(1 - e^2)}\right]^2 \cos i\,, \quad \delta\omega = +\,1°,194 \left[\frac{\mathrm{a}}{a(1 - e^2)}\right]^2 \left(1 - \frac{5}{4} \sin^2 i\right). \tag{12}$$

The ratio in the square brackets is but slightly less than one in case of artificial satellites.

Manifestly, the semi-major axis of the orbit and the inclination of the orbit plane do not undergo secular perturbation. The ascending node moves in a retrograde direction, while motion of the perigee is direct if $i < 63°26'$ and retrograde if $i > 63°26'$. Of course, periodic fluctuations may be superimposed on these variations, and under certain circumstances their amplitude may be considerable. From the expressions (11) the formula

$$\delta\omega + 2\,\delta\Omega = -\frac{36\pi\nu}{p^2}\sin^2\frac{i}{2}\left(1 - \frac{5}{3}\sin^2\frac{i}{2}\right)$$

can be derived instead of the classical approximation $\delta\omega + 2\delta\Omega = 0$ [7].

Still, a remark must be made on obtaining $\delta\omega$. The differential equations (8) and (10) yield a very large periodic perturbation for the trajectory element ω when excentricity is very small. This is simply due to the fact that the direction of the perigee is not sharply defined in an orbit that is almost circular. It is known that a theory simpler in principle and based on the smallness of e can be employed with advantage in this case. Results obtained for all other orbital elements are unaffected by the above circumstance.

No further integration is needed to compute δp or δe (which comes to the same thing if we know δa). The reason for this is as we know that the integral (7) can be expressed with orbital elements in the form

$$x\frac{dy}{dt} - y\frac{dx}{dt} = \sqrt{\mu p}\cos i$$

in case of Keplerian motion. This expression is strictly valid for any sort of disturbed motion, on account of the definition of osculating orbital elements but with a variable p and i. So in our case the relation

$$\sqrt{\mu p}\cos i = G = \text{const}$$

exists. Thus evidently p increases and decreases simultaneously with i, and since i undergoes no secular perturbation

$$\delta p = 0 \quad \text{and} \quad \delta e = 0. \tag{11*}$$

Having done with the first-order secular perturbations of the geometrical orbital elements let us turn to examining how this motion proceeds in time and let us ask how Kepler's 3-rd law is modified on account of the oblateness of the Earth. Let us consider the anomalistic period, reckoned from perigee to perigee, for example. In case of a slightly inclined orbit the gravitational attraction on the artificial satellite is always greater then μ/r^2 so its average angular velocity manifestly increases. On the other hand, the direction of the perigee advances, so the radius vector must describe an angle larger than 2π. In case of a large inclination of the orbit further complications arise. The question is, how are these effects to be added up.

Denoting the moment of a perigee transit by τ, the expression for the mean anomaly will be $M = n(t - \tau)$, where $n = \sqrt{\mu/a^3}$ is the mean motion pertaining to the momentary osculating trajectory; τ and n are of course functions of time. The total derivative of M with respect to time is

$$\frac{dM}{dt} = n + \frac{\delta M}{\delta t}, \quad \text{where} \quad \frac{\delta M}{\delta t} = -\sqrt{1-e^2}\left\{\frac{d\omega}{dt} + \cos i \frac{d\Omega}{dt} + \frac{2\,Sr}{\sqrt{\mu p}}\right\} \tag{13}$$

is the term arising from the variation of the orbital elements [8]. In the theory of special perturbations n is obtained by a quadrature, but in our case it is more convenient to resort to the energy-integral (6). For in case of Keplerian motion

$$\frac{1}{2}\left(\frac{d\mathbf{r}}{dt}\right)^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

so if we substitute $-\mu/2A$ for the energy-constant H, we obtain

$$\frac{1}{a} + \frac{2R}{\mu} = \frac{1}{A}.$$

A remains constant during the disturbed motion too; specifically at all perigee transits

$$\frac{1}{A} = \frac{1}{a_\tau}\left[1 + \frac{2\nu(1-3\sin^2\beta_\tau)}{a_\tau^2(1-e_\tau)^3}\right],$$

Thus

$$n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{\mu}{A^3}}\left[1 - \frac{2\nu A}{r^3}(1-3\sin^2\beta)\right]^{3/2} \approx \sqrt{\frac{\mu}{A^3}}\left[1 - \frac{3\nu a}{r^3}(1-3\sin^2\beta)\right]$$

and

$$\sqrt{\frac{\mu}{A^3}} \approx \sqrt{\frac{\mu}{a_\tau^3}}\left[1 + \frac{3\nu(1-3\sin^2\beta_\tau)}{a_\tau^2(1-e_\tau)^3}\right]$$

so the instantaneous value of the mean motion pertaining to the osculating trajectory is well approximated by

$$n = n_\tau\left\{1 + \frac{3\nu(1-3\sin^2\beta_\tau)}{a_\tau^2(1-e_\tau)^3} - \frac{3\nu a}{r^3}(1-3\sin^2\beta)\right\}. \tag{14}$$

Seeking to obtain the deviation of the disturbed mean anomaly from the undisturbed, we shall integrate expressions (13) and (14) over the interval between two perigee transits. Since we are dealing with first-order perturbations, it is indifferent in case of the terms containing ν whether integration is performed over the period $P = 2\pi/n_\tau = 2\pi\sqrt{a_\tau^3/\mu}$ of Keplerian motion, or over the anomalistic period P_π of disturbed motion that we are seeking. Likewise, the orbital elements may be regarded as constant. On these grounds, and using (9):

$$n = n_\tau\left\{1 + \frac{3\nu(1-3\sin^2\beta_\tau)}{a_\tau^2(1-e_\tau)^3}\right\} + \sqrt{1-e^2}\,\frac{Sr}{\sqrt{\mu p}}$$

and the integral in question is

$$2\pi = n_\tau \left\{ 1 + \frac{3\nu(1-3\sin^2\beta_\tau)}{a_\tau^2(1-e_\tau)^3} \right\} P_\pi - \sqrt{1-e^2} \left\{ \delta\omega + \cos i\,\delta\Omega + \frac{1}{\sqrt{\mu p}} \int_\tau^{\tau+P} Sr\,dt \right\}.$$

But, similarly to the deduction of (11)

$$\frac{1}{\sqrt{\mu p}} \int_\tau^{\tau+P} Sr\,dt = \frac{1}{\mu p} \int_0^{2\pi} Sr^3\,dv = -\frac{3\nu}{p^2} \int_0^{2\pi} (1-3\sin^2 i \sin^2 u)(1+e\cos v)\,dv =$$

$$= -\frac{6\pi\nu}{p^2}\left(1-\frac{3}{2}\sin^2 i\right) = -(\delta\omega + \cos i\,\delta\Omega)$$

So the anomalistic period is

$$P_\pi = \frac{2\pi}{n_\tau}\left\{1 - \frac{3\nu(1-3\sin^2\beta_\tau)}{a_\tau^2(1-e_\tau)^3}\right\}$$

or taking into account the facts that $\mu = fm_E$ and that $\sin\beta_\tau = \sin i \sin\omega$, for the latitude of the direction of the perigee, we find

$$P_\pi = 2\pi \sqrt{\frac{a^3}{fm_E}} \left\{ 1 - \frac{3\nu}{a^2(1-e)^3}(1-3\sin^2 i \sin^2\omega) \right\} \tag{15}$$

having dropped the now unnecessary index τ.

Plainly P_π is shorter than the Keplerian period if the latitude of the perigaeum $< 35°16'$, and longer if it is $> 35°16'$. Furthermore, owing to the secular variation of ω, P_π is by no means constant, rather does it oscillate periodically around the mean

$$\overline{P}_\pi = \frac{1}{2\pi}\int_0^{2\pi} P_\pi\,d\omega = 2\pi\sqrt{\frac{a^3}{fm_E}}\left\{1 - \frac{3\nu}{a^2(1-e)^3}\left(1-\frac{3}{2}\sin^2 i\right)\right\}.$$

Using the constant $fm_E = 3{,}986329.\ 10^{20}\ \mathrm{cm}^3\,\mathrm{sec}^{-2}$ [3] we obtain the numerical expression

$$\overline{P}_\pi = 84^m\,29^s{,}4\left(\frac{a}{\mathrm{a}}\right)^{3/2} - 8^s{,}4\left(\frac{a}{\mathrm{a}}\right)^{-1/2}\frac{1-1{,}5\cdot\sin^2 i}{(1-e)^3}.$$

We have shown that those main characteristics of the motion of artificial satellites that are due to purely gravitational forces, summarized in the results of (11), (11*), and (15), can be obtained in a comparativelly elementary way. For a detailed analysis of the observations, however, second-order perturbations must at least be taken into account, the theory of which requires one further term to be added to the force-function (1). In the case of artificial satellites revolving at a great height probably the influence of the Moon and the Sun may not be entirely negligible either. In order to decide these questions, more sophisticated methods of celestial mechanics are needed.

Thus it is significant that the Hill-Brown lunar theory has been applied in relation to our problem. Second-order theories have been submitted by D. Brouwer [9], and A. A. Orlov [10], for the cases of slight excentricity and slight inclination of the orbit, and slight excentricity and arbitrary inclination of the orbit, respectively. For the case of arbitrary excentricity and arbitrary inclination of the orbit the theory of A. A. Orlov [11] can only be regarded as a first-order one. For the choice of the intermediary orbit the essay of T. E. Sterne [12] provides novel points of view.

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