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THE DIFFERENTIAL ROTATION AND THE LARGE-SCALE MERIDIONAL MOTION OF THE STARS

BY

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MERIDIONÁLIS ÁRAMLÁS A FORGÓ CSILLAGOKBAN

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Jelen dolgozat a csillagok tengelyforgásával foglalkozik, figyelembe véve a meridionális áramlásokat, de elhanyagolva az elektromágneses hatásokat. Feltételezve azt, hogy a csillag turbulens állapotban van, a szerző az adiabatikus állapotegyenletet alkalmazta. A meridionális áramlás sebességének vektorpotenciálját, a tengelyforgás sebességét és ennek örvényeit (rotacióját) differenciálegyenletek határozzák meg. Ezek megoldását a Fourier-elv alapján gömbfüggvények végtelen sorába fejtettük. A megvitatásból az következik, hogy a meridionális áramlás sebessége független adat, melyet a többi állandóval együtt (a csillag sugara, tengelyforgás sebessége stb.) kell megadnunk. Ez a következtetés azonban csak akkor helyes, ha elhanyagoljuk az elektromágneses hatásokat. A csillag tengelyforgása és mágneses tere közötti kapcsolatból arra következtethetünk, hogy a meridionális áramlás sebességét a mágneses tér erősségéből lehet levezetni.

THE DIFFERENTIAL ROTATION AND THE LARGE-SCALE MERIDIONAL MOTION OF THE STARS.

By *I. K. Csada.*

Extract. The present paper contains a discussion on the rotation of the stars considering the meridional currents but neglecting the electromagnetic effects. Assuming that the whole interior of the star is in turbulent state, the author applies the adiabatical equation of state. Differential equations are deduced for the vector potential producting the velocity of the meridional current and differential equations are deduced for rotational velocity and its vorticity. Their approximate solutions are reached by a series of spherical harmonics. As it is shown in the discussion, if we consider hydrodynamical effects only, the velocity of the meridional current is to be given together with the stellar structure. We have to conclude, that after having considered the electromagnetic effects, the velocity of the meridional current can be deduced from the strength of the magnetic-field, respectively from the angular velocity, if we suppose a relation existing between the last two.

1. **Introduction.** The axial rotation of the sun and stars, as observations confirm it, differs essentially from that of a rigid body. The vorticity of the velocity (curl of velocity), one of the most characteristic dates concerning the state of motion of a rotating rigid body, is a constant in direction and magnitude everywhere in space. This quantity is in the case of the sun and probably of all rotating stars a general function of space. Observations of the rotation of the sun's surface give us some idea about the distribution of this function. The angular velocity (i. e. the vorticity of the velocity) is in the polar areas considerably less than along the equator. Accordingly, the rotation of the stars is not characterised by the boundary condition of rigid bodies, where the adjacent particles cannot change their relative positions. Describing more generally the axial rotation, we assume, that neither direction nor magnitude of the vorticity of the velocity-field is constant. Its function of distribution is determined by mechanical, thermodynamical and electrodynamical conditions. However Euler's hydrodynamical equations are not suitable for the general discussion of the problem. We have to complete them with viscosity terms and we have to consider the electromagnetic forces too. Employing the turbulence theory the former problem can be solved, but considering the latter we encounter difficulties of principle. Namely, the reciprocal influences of the pure mechanical and electrodynamical forces are at present quite problematic. It is certain, that some relation exists between rotation and magnetism

of the stars, but until now no satisfactory theory can explain it. *Alfvén*[1] in his magneto-hydrodynamical theory assumes besides the motion of neutral particles also that of electrically charged particles (thermoelectric currents) and deduces hereby relations between rotating and magnetic phenomena. According to *Blackett*'s theory[2] magnetic space is also generated by neutral mass-flux if it derives from rotation. However this new theory has only a very weak empirical basis. On the whole it leads to the completion of the hydrodynamical equations with Lorentz's law of forces and Maxwell's equations. As a result of these we arrive at *Alfvén*'s magneto-hydrodynamics, which may thus be considered the latest form of drawing up this problem in the mathematical sense.

In the present paper we examine only the mechanical conditions of the rotation assuming that viscosity originates from the turbulent state.

2. Fundamental equations. Sketching out in general the axial rotation of the stars we assume that vorticity of the velocity is a general function of space. Let $\mathfrak{v}$ be the velocity and $\mathfrak{v}_r$, $\mathfrak{v}_\vartheta$, $\mathfrak{v}_\varphi$ its polar coordinates of space. Let us suppose that the system is independent of the coordinate φ, in that case we get for the polar coordinates of the vorticity :

$$
\begin{aligned}
\mathfrak{c}_r &= \frac{1}{r \sin\vartheta} \frac{\partial}{\partial \vartheta} (\sin\vartheta . \mathfrak{v}_\varphi) \\
\mathfrak{c}_\vartheta &= -\frac{1}{r} \frac{\partial}{\partial r} (r\, \mathfrak{v}_\varphi) \\
\mathfrak{c}_\varphi &= \frac{1}{r} \frac{\partial}{\partial r} (r\, \mathfrak{v}_\vartheta) - \frac{1}{r} \frac{\partial \mathfrak{v}_r}{\partial \vartheta}
\end{aligned}
\tag{1}
$$

where corresponding to rot $\mathfrak{v} = \mathfrak{c}$, the components of the vorticity (of the velocity) are $\mathfrak{c}_r$, $\mathfrak{c}_\vartheta$, $\mathfrak{c}_\varphi$. In the case of axial rotation of a rigid body the vorticity is parallel to the rotational axis, therefore, its φ-component being perpendicular to the meridian plane is identically zero. On the other hand from (1) we can immediatelly find out that this is only possible in the case when the velocity has a component only in the meridian plane (r and ϑ components are zero). In the case of the stars, the velocity has components in the direction of all three coordinates. Consequently, to the former (circulatory) motion is added another motion which occurs only in the meridional plane (meridional current). The vorticity of the meridional current has a φ component only.

In the following, our object is to express $\mathfrak{v}$ explicitly. We take as bases the hydrodynamical equations of motion and the equation of continuity, so we can write in vectorial form :

$$
\mathfrak{v}\, \mathfrak{grad}\, \mathfrak{v} = \text{grad}\, V - \frac{1}{\varrho}\, \text{grad}\, P - \frac{A}{\varrho}\, \text{rot rot}\, \mathfrak{v} \tag{2}
$$

[1] Arkiv för Matematik, Astronomi och Fysik Bd. 29 A, No. : 11, 12 ; Bd. 29 B, No. 2.

[2] Nature, 159, 658 (1947).

$$\operatorname{div} \varrho \mathfrak{v} = 0. \tag{3}$$

For future purposes we need also the equation of the conservation of energy, however, similarly to the equations of motion we have to transform it for the case of the turbulent state:

$$\frac{\partial}{\partial t} (\varrho E) + \operatorname{div} (\mathfrak{F} + E \varrho \mathfrak{v}) + P \operatorname{div} \mathfrak{v} = 0. \tag{4}$$

Further we need Poisson's equation of gravitational potential:

$$\Delta V = -4 \pi G \varrho \tag{5}$$

the equation of state and other equations concerning the internal structure.

The system being independent of φ only the r- and ϑ- velocity components producing meridional currents appear in (3). Let us denote the resultant of these two components (namely the velocity of the meridional current) with $\mathfrak{w}$ so the equation of continuity takes the following form:

$$\operatorname{div} \varrho \mathfrak{w} = 0. \tag{6}$$

From this equation it follows in general that $\varrho \mathfrak{w}$ can be expressed by vector potential:

$$\varrho \mathfrak{v} = \operatorname{rot} \mathfrak{A} \tag{7}$$

Taking its rotation-operator (curl) we obtain a differential equation for the vector potential:

$$\Delta \mathfrak{A} = -\operatorname{rot} \varrho \mathfrak{w} \tag{8}$$

Transforming the right-hand side in accordance with a known vector-analytical formula and employing the equation (7) we have

$$\Delta \mathfrak{A} + [\operatorname{grad} \log \varrho, \operatorname{rot} \mathfrak{A}] = -\varrho \operatorname{rot} \mathfrak{w} \tag{9}$$

This vectorial equation gives in fact only one scalar equation for, since components r and ϑ of $\mathfrak{A}$ and rot $\mathfrak{w}$ are both identically zero. Making use of the relation rot $\mathfrak{w} = \mathfrak{c}$ we obtain:

$$\Delta \mathfrak{A}_\varphi - \frac{\mathfrak{A}_\varphi}{r^2 \sin^2 \vartheta} - \frac{\partial \log \varrho}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r \mathfrak{A}_\varphi) + \frac{1}{r^2} \frac{\partial \log \varrho}{\partial \vartheta} \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} (\sin \vartheta \mathfrak{A}_\varphi) = -\varrho \mathfrak{c}_\varphi.$$

To the solution we need $\mathfrak{c}_\varphi$ as a function of the coordinates, which we shall derive from the equation of motion in the next paragraph.

3. **Mathematical development of the vorticity of the velocity.** Let us take the rot of (2):

$$\operatorname{rot} (\mathfrak{v} \operatorname{grad} \mathfrak{v}) = -\left[\operatorname{grad} \frac{1}{\varrho}, \operatorname{grad} P\right] + \frac{A}{\varrho} \Delta \mathfrak{c}. \tag{10}$$

We can calculate $\mathfrak{c}_\varphi$ from this equation if we know the left-hand side and the vectorial product on the right-hand side. To get $\mathfrak{A}_\varphi$ we need only the φ component of $\mathfrak{c}$, which means that we need only the

φ component of the vectorial equation (10). In fact, in this case we use only the r and ϑ components of (2) to obtain the rotation.

The equation (10) is analogous to the equation deduced first by *Bjerkness* in 1921 while discussing the problem of rotation.[1] If we neglect the viscosity terms and the velocity of the meridional current as small compared with the velocity of the rotational motion, we get (10) in cylindrical coordinates :

$$2\,\mathfrak{v}_z \frac{\partial \omega}{\partial z} = [\mathrm{grad}\frac{1}{\varrho}\,,\ \mathrm{grad}\,P]$$

where the angular velocity $\mathfrak{w}$ is a function of space. In his theory of the meridional current *Randers* also used this equation as basis.[2] He was the first to prove that the vectorial product on the left-hand side cannot be zero. Namely, the angular velocity in this case is a function of the distance from the axis only, but this leads to very rigorous mechanical conditions and excludes the existence of the meridional current. The difficulty arises when the vectorial product on the left is zero, as in all such models where the pressure is only a function of density (polytropic and adiabatic models). Eddington's model, which describes the internal constitution of the stars in the first approximation, is one of them too. We have to suppose that pressure cannot be represented as a function of density only ; it has to contain the temperature too.

In the case of the turbulence state A/ϱ is a very large number therefore the last term of (10) cannot be neglected. On the other hand, the adiabatic relation between pressure and density is one of the conditions for the turbulent (convective) state. Although this concerns strictly non-rotating stars only, it still justifies us in neglecting the vectorial product instead of the last term.

We shall neglect certain terms on the right-hand side of (10) too. Making use of a known vector-analytical formula, we have :

$$\mathfrak{v}\,\mathfrak{grad}\,\mathfrak{v} = -[\mathfrak{v},\ \mathrm{rot}\,\mathfrak{v}] + \frac{1}{2}\,\mathrm{grad}\,\mathfrak{v}^2$$

On the other hand :

$$\mathfrak{v} = \mathfrak{v}_\varphi + \mathfrak{w}$$

and

$$\mathrm{rot}\,\mathfrak{v} = \mathrm{rot}\,\mathfrak{v}_\varphi + \mathrm{rot}\,\mathfrak{w} = \mathfrak{c}_m + \mathfrak{c}_\varphi$$

thus we have :

$$[\mathfrak{v},\ \mathrm{rot}\,\mathfrak{v}] = [\mathfrak{v}_\varphi,\ \mathfrak{c}_m] + [\mathfrak{v}_\varphi,\ \mathfrak{c}_\varphi] + [\mathfrak{w},\ \mathfrak{c}_m] + [\mathfrak{w},\ \mathfrak{c}_\varphi]$$

$\mathfrak{v}_\varphi$ and $\mathfrak{c}_\varphi$ are parallel, therefore, the second term is zero, the fourth contains the velocity of the meridional current in second order only. Dropping them we get for the components of $[\mathfrak{v},\ \mathrm{rot}\,\mathfrak{v}]$:

[1] Geofysiske Publikasjoner II. Nr. 4. 1921.
[2] Ap. J. 94, 109 (1941).

$$[\mathfrak{v}, \operatorname{rot} \mathfrak{v}]_r = [\mathfrak{v}_\varphi, \mathfrak{c}_m]_r = \frac{\mathfrak{v}_\varphi^2}{r} + \frac{1}{2} \frac{\partial \mathfrak{v}_\varphi^2}{\partial r}$$

$$[\mathfrak{v}, \operatorname{rot} \mathfrak{v}]_\vartheta = [\mathfrak{v}_\varphi; \mathfrak{c}_m]_\vartheta = \frac{\mathfrak{v}_\varphi^2}{r \operatorname{tg} \vartheta} + \frac{1}{2} \frac{1}{r} \frac{\partial \mathfrak{v}_\varphi^2}{\partial \vartheta}.$$

$$[\mathfrak{v}, \operatorname{rot} \mathfrak{v}]_\varphi = [\mathfrak{w}, \mathfrak{c}_m]_\varphi.$$

Making use of these the φ component of the left-hand side of (10) is:

$$\operatorname{rot}_\varphi (\mathfrak{v} \operatorname{grad} \mathfrak{v}) = -\frac{1}{r \operatorname{tg} \vartheta} \frac{\partial \mathfrak{v}_\varphi^2}{\partial r} + \frac{1}{r^2} \frac{\partial \mathfrak{v}_\varphi^2}{\partial \vartheta}.$$

Let us express $\mathfrak{v}_\varphi$ with the angular velocity

$$\mathfrak{v}_\varphi = \omega r \sin \vartheta$$

so

$$\operatorname{rot}_\varphi (\mathfrak{v} \operatorname{grad} \mathfrak{v}) = -r \frac{\partial \omega^2}{\partial r} \sin \vartheta \cos \vartheta + \frac{\partial \omega^2}{\partial \vartheta} \sin^2 \vartheta$$

and the φ component of (10):

$$\frac{A}{\varrho} \left(\Delta \mathfrak{c}_\varphi - \frac{\mathfrak{c}_\varphi}{r^2 \sin^2 \vartheta} \right) = r \frac{\partial \omega^2}{\partial \vartheta} \sin \vartheta \cos \vartheta - \frac{\partial \omega^2}{\partial \vartheta} \sin \vartheta. \tag{11}$$

If the angular velocity is constant, the right-hand side is zero and we can solve the equation easily:

$$\mathfrak{c}_\varphi = \sum_{n=0}^{\infty} C_n P_n^{(1)} r^n \tag{12}$$

where C_0, C_1, C_2 are arbitrary constants. The solution however must be symmetrical to the equator, which means that n is an even number. So the simplest solution of (10) if $n = 2$:

$$\mathfrak{c}_\varphi = C_2 r^2 P_2^{(1)}.$$

In the general case only symmetry and conditions of continuity set limits for the solution of (12). These conditions are fulfilled in the interpolation formulae of the rotation of the sun's surface, at least for the coordinate ϑ, and therefore it is reasonable to choose them for starting. The interpolation formulae may generally be written as:

$$\omega = \omega_0 + \omega_2 \cos^2 \vartheta + \omega_4 \cos^4 \vartheta + \ldots$$

where ω_0, ω_2 are constants. But ω is a function of r too, which means that ω_0, ω_2, ... must be functions of r. Let us suppose that we can expand ω in the following series:

$$\omega = \frac{A}{\varrho} \sum_{r=0}^{\infty} w_n \cos^n \vartheta \tag{13}$$

where w_n is a function of r and n runs over the even numbers. By squaring (13) we get the following series:

$$\omega^2 = \left(\frac{A}{\varrho}\right)^2 \sum_{n=0}^{\infty} b_n \cos^n \vartheta \tag{14}$$

where

$$b_n = \sum_{k=0}^{n} w_k \; w_{n-k}.$$

Let us insert (14) into the right-hand side of (11):

$$\Delta\, c_\varphi - \frac{c_\varphi}{r^2 \sin^2 \vartheta} = \sum_{n=0}^{\infty} \frac{A}{\varrho}\left[\left(nb_n + r\frac{\partial b_n}{\partial r}\right) \cos^{n+1}\vartheta \sin\vartheta + nb_n \cos^{n-1}\vartheta \sin\vartheta\right].$$

We can transform the right-hand side simply into a series arranged in terms of Legendre's associated functions. In fact we use the following formulae:

$$\sin\vartheta = P_1^{(1)}$$

$$\cos\vartheta \;\sin\vartheta = \frac{1}{3} P_2^{(1)}$$

$$\cos^2\vartheta \;\sin\vartheta = \frac{1}{5} P_1^{(1)} + \frac{2}{3} P_3^{(1)}$$

$$\cos^3\vartheta \;\sin\vartheta = \frac{1}{7} P_3^{(1)} + \frac{2}{35} P_4^{(1)}.$$

Thus we obtain from (12):

$$\Delta\, c_\varphi - \frac{c_\varphi}{r^2 \sin^2\vartheta} = \sum_{n=0}^{\infty} \frac{A}{\varrho} f_n(r)\, P_n^{(1)} \tag{15}$$

where

$$f_2(r) = \frac{8}{21} b_2 + \frac{16}{63} b_4 + \ldots + r\frac{d}{dr}\left(\frac{1}{3} b_0 + \frac{1}{7} b_2 + \frac{5}{35} b_4 + \ldots\right)$$

($f_n(r) = 0$, if n is an odd number).

Equation (15) may be solved by the following series:

$$c_\varphi = \sum_{n=0}^{\infty} \frac{A}{\varrho}\, \frac{C_n}{r}\, P_n^{(1)} \tag{16}$$

where n runs over the odd numbers only. C_n is a function of r and can be obtained from the following differential equation:

$$C_n'' - \frac{n\,(n+r)}{r^2} C_n = r\, f_{n+1}. \tag{17}$$

We get the solution by using the method of variation of parameters. Solving the reduced equation derived from (17) we get r^{n+1} and r^{-n} and so the complete solution free of singularity is:[1]

$$C_n = -r^{n+1} \int_0^r \frac{r^{-n+1} f_{n+1}}{\Delta\, (r^{n+1},\, r^{-n})}\, dr$$

where $\Delta\ (r^{n+1}, r^{-n})$ is the Wronskian of the two particular solutions:

$$\Delta\ (r^{n+1},\, r^{-n}) = -2n-1$$

hence

$$C_n = \frac{r^{n+1}}{2n+1} \int_0^r \frac{f_{n+1}}{r^{n-1}}\, d\, r. \tag{18}$$

Calculating the vorticity, it is a question of great importance whether in the stellar interior c_φ has a constant sign or not. If the sign of c_φ is constant, the meridional current forms a single large-scale vortex-ring in the hemisphere, on the other hand, if its sign changes the current forms one or perhaps more systems of separate vortex-rings circulating in different directions. The formula (18) expresses only the existence of a relationship between c_φ and ω, therefore, we have to find further equations of condition for c_φ as well as for ω. Having examined all conditions we have to decide purely theoretically whether c_φ changes its sign or not.

4. **Mathematical development of the distribution of pressure, density and temperature.** It is known that the distribution of pressure, density and temperature of a star undergoes a change already in the case of uniform rotation. The change is easily explained by the centrifugal force. The mathematical apparatus to obtain the function of the distribution is also well known for the case of politropic gas-spheres. Essentially the same apparatus is used in the case of differential rotation, if we neglect the meridional currents. In case of meridional currents, however, thermodynamical processes (heat-transport) appear too, simultaneously with pure mechanical processes (transport of momentum or vorticity). In the case of such currents energy is transported not only by radiation and (turbulent) convection, but by large-scale currents too. The distribution of temperature will not be rotationally symmetrical, but the temperature will be higher in the area of ascending currents (polar areas), and lower in that of descending currents. Density and pressure change similarly.

We shall start in our mathematical discussion from the equation of the energy. One of the conditions for a meridional current is the very high internal friction which can be explained only by the turbulent state of the star. So we have to alter the equation of conservation of energy in conformity whith the „mixing length" hypothesis in the same manner as

[1] Ince: Ordinary differential equations (1926) p.: 122.

we have done with the mechanical equations. The equation of the energy for turbulent state will take the form :[1]

$$\varrho \frac{dE}{dt} - \frac{P}{\varrho} \frac{d\varrho}{dt} + \operatorname{div}\left\{\mathfrak{F} + c_p A \left[\left(\frac{dT}{dP}\right)_{ad} \operatorname{grad} P - \operatorname{grad} T\right]\right\} - \tag{19}$$

$$- \left(\frac{A}{\varrho} \operatorname{grad} V, \left(\frac{d\varrho}{dP}\right)_{ad} \operatorname{grad} P - \operatorname{grad} \varrho\right) = 0.$$

On the other hand, one sufficient condition of the aimed convective state is the adiabatic relation between pressure, density and temperature, so we have :

$$T\varrho^{1-\gamma} = c_1, \qquad P\varrho^{-\gamma} = \frac{k}{\mu H} c_1, \qquad P^{1-\gamma} T^{\gamma} = \left(\frac{k}{\mu H}\right)^{1-\gamma} c_1$$

where $\gamma = c_r/c_v$. It is easily seen that in this case the quantities in the brackets [ ] are equal to zero and so (19) will be :

$$\varrho \frac{dE}{dt} - \frac{P}{\varrho} \frac{d\varrho}{dt} + \operatorname{div} \mathfrak{F} = 0. \tag{20}$$

In the case of stationary energy production $\partial E/\partial t$ is not zero, but it is a function of space only ; we write :

$$\frac{\partial E}{\partial t} = \varepsilon\ (r, \vartheta).$$

If the mechanical state is also stationary, $\partial\varrho/\partial t = 0$ and the equation of energy is :

$$\varrho\, \varepsilon + (\varrho\, \mathfrak{v}, \operatorname{grad} E) - \left(\frac{P}{\varrho}\ \mathfrak{v}, \operatorname{grad} \varrho\right) + \operatorname{div} \mathfrak{F} = 0. \tag{21}$$

If we neglect the ionization energy, we get

$$E = \frac{3}{2} \frac{k}{\mu H} T + \frac{a T^4}{\varrho}$$

or making use of the adiabatic relations :

$$E = \frac{3}{2} \frac{k}{\mu H} c_1 \varrho^{\gamma-1} + a\, c_1^4 \varrho^{4\gamma-5}.$$

The radiation-flux $\mathfrak{F}$ is proportional to the gradient of the temperature :

$$\mathfrak{F} = -c \operatorname{grad} T$$

where c is the conductivity. Let us insert these formulae into (21) :

$$\varrho\, \varepsilon + \left(\left\{\frac{3\gamma-5}{2} \frac{k}{\mu H} c_1 \varrho^{\gamma-1} + (4\gamma-5)\, a\, c_1^4 \varrho^{4\gamma-5}\right\} \mathfrak{v}, \operatorname{grad} \varrho\right) = \operatorname{div} c \operatorname{grad} T$$

[1] Astrophysica Norvegica 4, 39 [1946].

The critical value of γ is 4/3 when the left-hand side of the scalar product is nearly equal zero. For $\gamma = 5/3$ the first, for $\gamma = 5/4$ the second member equals zero. The first value is positive, the second negative. For stars γ equals cca 1·5 therefore the whole expression may be neglected in the first approximation. Further let us assume c constant in first approximation then the equation of energy becomes, also in the case of turbulent state :

$$-\frac{\varrho\,\varepsilon}{c} = \Delta T. \tag{23}$$

In fact we determine the distribution of density, pressure and temperature from the equation of motion by eliminating the gravitational potential. *Rosseland*[1] using Poisson's equation and the equation of the conservation of energy expressed the potential for the standard model as a function of T only and thus he obtained a general differential equation for the distribution of density in a short and elegant way. However, if c and ε are not constant this method can not be applied and it is more convenient to form the divergence of (2), eliminating the potential directly by means of the Poisson equation :

$$\text{div}\,(\mathfrak{v}\,\mathfrak{grad}\,\mathfrak{v}) = -4\pi G\varrho - \text{div}\frac{1}{\varrho}\,\text{grad}\,P. \tag{24}$$

But this equation can be solved only if we know the distribution of the velocity. The function of the distribution thus derived has to satisfy another condition too imposed by (23) (i. e. heat has to be transported too, simultaneously with the mass-flux).

Using (22) and (23) we can express ϱ as a function of the distribution of velocity and with it we are able to eliminate ϱ from the differential equation (8) of the vector potential. Let us express P in (24) with the help of the adiabatic relation as a function of T:

$$\text{div}\,\frac{1}{\varrho}\,\text{grad}\,P = \frac{\gamma}{\gamma-1}\,\frac{k}{\mu H}\,\Delta T$$

Using (23) we may eliminate ΔT and hence (24) takes the form :

$$\text{div}(\mathfrak{v}\;\mathfrak{grad}\mathfrak{v}) = -\left(4\pi G - \frac{\gamma}{\gamma-1}\,\frac{k}{\mu H}\,\frac{\varepsilon\,c_1}{c}\right)\varrho. \tag{25}$$

We get the right-hand side in the same way as in § 2 (we neglect the products of second order in the meridional current's velocity) :

$$\mathfrak{v}\;\mathfrak{grad}\;\mathfrak{v} \approx \left(-\frac{\mathfrak{v}_\varphi}{r},\,-\frac{\mathfrak{v}_\varphi}{r\,tg\vartheta},\,0\right)$$

whence

$$\text{div}(\mathfrak{v}\;\mathfrak{grad}\mathfrak{v}) = -\frac{1}{r^2}\,\frac{\partial}{\partial r}\,(r\,\mathfrak{v}_\varphi^2) - \frac{1}{r^2\sin\vartheta}\,\frac{\partial}{\partial\vartheta}\,(\cos\vartheta\;\mathfrak{v}_\varphi)$$

[1] Astrophysica Norvegica 2, 173; 249 (1937).

Let us substitute $\mathfrak{v}_\varphi = \omega r \sin\vartheta$:

$$\operatorname{div}(\mathfrak{v}\,\mathrm{grad}\,\mathfrak{v}) = -2\omega^2 - r\frac{\partial\omega^2}{\partial r}\sin^2\vartheta - \frac{\partial\omega^2}{\partial\vartheta}\sin\vartheta\cos\vartheta$$

so (25) takes the form:

$$a\varrho = 2\omega^2 + r\frac{\partial\omega^2}{\partial r}\sin^2\vartheta - \frac{\partial\omega^2}{\partial\vartheta}\sin\vartheta\cos\vartheta \tag{26}$$

where

$$a = 4\pi G - \frac{\gamma}{\gamma-1}\frac{c_1\varepsilon}{c}.$$

If we do not neglect the terms of higher order in $\mathfrak{v}$ grad$\mathfrak{v}$ the right-hand side contains also $\mathfrak{v}_\varphi$ and $\mathfrak{v}_\varphi$ beside ω^2. Hence there is a complicated relation between density and the velocities which in the first approximation, after neglecting the terms containing the derivatives takes the form:

$$\varrho = \frac{2\omega^2}{a}. \tag{26. a}$$

On the stellar suface the value of ϱ is equal to zero, consequently ω must be zero too, on the other hand if instead of neglecting the terms containing derivatives we put in their average valuesthe following term will be added to the equation:

$$M_\omega = \frac{3}{4r^3\pi}\iiint\left(r\frac{\partial\omega^2}{\partial r}\sin^2\vartheta - \frac{\partial\omega^2}{\partial\vartheta}\sin\vartheta\cos\vartheta\right) r^2\sin\vartheta\, dr\, d\vartheta\, d\varphi.$$

Integrating with respect to ϑ and φ (putting $\omega^2 = \Sigma\left(\frac{A}{\varrho}\right)^2 b_n\cos^n\vartheta$)

we get:

$$M_\omega = \sum\frac{6}{(n+1)\,(n+3)}\left(b_n + \frac{n-3}{r_0^3}\int_0^{r_0} r^2\, b_n\, dr\right)$$

which cannot be identically zero. So for (26. a) we get:

$$\varrho = M_\omega + \frac{2\omega^2}{\alpha}. \tag{26. b}$$

5. **Mathematical development of the vector potential.** No difficulty is encountered in calculating the vector potential $\mathfrak{A}$, on the basis of (9) if we know $\mathfrak{c}_\varphi$. The differential equation for the vector potential is:

$$\Delta\,\mathfrak{A}_\varphi - \frac{\partial\log\varrho}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(r\,\mathfrak{A}_\varphi) + \frac{1}{r^2}\frac{\partial\log\varrho}{\partial\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathfrak{A}_\varphi) -$$
$$- \frac{\mathfrak{A}_\varphi}{r^2\sin^2\vartheta} = -\varrho\,\mathfrak{c}_\varphi. \tag{27}$$

We may eliminate ϱ by making use of (26). To carry out this calculation would take rather a long time, in approximation we omit from (26) as well as from (27) the terms containing differential quotients of first order and thus the differential equation for the vector potential will take the form:

$$\Delta \mathfrak{A}_{\varphi} - \frac{\mathfrak{A}_{\varphi}}{r^2 \sin^2\vartheta} = - \varrho\, \mathfrak{c}_{\varphi}. \tag{28}$$

The solution will be:

$$\mathfrak{A}_{\varphi} = - \sum_{n=0}^{\infty} \frac{A}{\varrho} \frac{A_n}{r} P_n^{(1)} \tag{29}$$

and we write:

$$\varrho\, \mathfrak{c}_{\varphi} = \sum_{n=0}^{\infty} B_n P_n^{(1)} \tag{30}$$

where making use of $\varrho = \Sigma \varrho_n P_n$ and $\mathfrak{c}_{\varphi} = \Sigma \frac{A}{\varrho} \frac{C_n}{r} P_n^{(1)}$ we get:

$$B_1 = \frac{c_1}{r} \left(\varrho_0 - \frac{1}{5} \varrho_2\right)$$

$$B_2 = \frac{c_3}{r} \varrho_0 + \frac{1}{5} \frac{c_1}{r} \varrho_2.$$

Putting (29) and (30) into (28) we get the following differential equation for A_n:

$$A_n'' - \frac{n(n+1)}{r^2} A_n = B_n \tag{31}$$

the solution (in the case of an infinitesimal rotation) will be:

$$A_n = \sum_k a_{nk}\, \varphi_{nk} (\varkappa_{nk} r).$$

Let

$$B_n = \sum b_{nk}\, \varphi_{nk} (\varkappa_{nk} r)$$

where we have the following differential equation for φ_{nk}:

$$\varphi_{nk}'' + \left(\varkappa_{nk}^2 - \frac{n(n+1)}{r^2}\right) \varphi_{nk} = 0 \tag{32}$$

and for the coefficients:

$$\varkappa_{nk}^2\, a_{nk} = b_{nk}. \tag{33}$$

6. Boundary conditions. It is a general characteristic of all rotating stars that the velocity-field has no component perpendicular to the surface of the star. This imposes a condition upon the vector potential of the velocity, which can be fulfilled only for certain values of the constants in the expression of the potential. It was shown by *Randers* that the condition so imposed upon the velocity-field is equivalent with the condition that the potential on the boundary of the meridional quadrant equals zero. Consequently φ_{nk} will be eigenfunction of the problem and $\varkappa_{nk}$ eigenvalue.

For the lowest eigenvalues $\varkappa_{no}$ the eigenfunctions φ_{no} have a zero value only on the boundary of the meridional quadrant. In this case the meridional current forms a single large-scale vortex-ring. For the second eigenvalue $\varkappa_{n2}$ has a zero value also inside the boundary of the meridional quadrant. Therefore, the hemisphere splits in two vortex-rings in which circulation runs in consequence of the vorticity in opposite directions. In the case of higher eigenvalues there are several, separate vortex-rings each circulating in the opposite direction to the next.

The boundary line of the meridional quadrant runs partly over the stellar surface. So the boundary conditions depend on the occasional form of the star. We have already mentioned in the foregoing that the form of the star undergoes a change in consequence of the rotation. Therefore, we can take into consideration the boundary conditions only after having previously solved the problem. In the case of an infinitesimal rotation it is easily seen that it is sufficient to apply the boundary condition to the undistorted sphere.

7. The velocity of the meridional current. On the basis of (6) using (20) and (23) we can express the velocity of the meridional current with the functions φ_{nk} and P_n:

$$\varrho\, \mathfrak{v}_r = \frac{A}{\varrho} \sum_n \sum_k \frac{b_{nk}}{\varkappa_{nk}^2} \frac{\varphi_{nk}}{r^2}\, n\, (n+1)\, P_n \tag{34}$$

$$\varrho\, \mathfrak{v}_\vartheta = \frac{A}{\varrho} \sum_n \sum_k \frac{b_{nk}}{\varkappa_{nk}^2} \frac{1}{r} \frac{d\, \varphi_{nk}}{dr} P_n^{(1)}.$$

We do not know yet the coefficient b_{nk} in these expressions. To obtain it we use the φ component of the equation of motion, exhausting therewith the whole mechanical equation-stock of the problem. Of the first two equations of (2) we have so far used only one relation to calculate the vorticity, but we have to develop the distribution of density also from the mechanical equations. To get it we have to form the divergence of the equation of motion (2) and making use of Poisson's equation of the gravitational potential and of the equation of state, we obtain a differential equation for ϱ.

8. The determination of the φ component of velocity. The φ component of the equation of motion is:

$$(\mathfrak{w},\ \operatorname{grad} \mathfrak{v}_\varphi) + \left(\mathfrak{v}_r + \frac{\mathfrak{v}_\vartheta}{tg\,\vartheta}\right) \frac{\mathfrak{v}_\varphi}{r} = \frac{A}{\varrho} \left(\Delta\, \mathfrak{v}_\varphi - \frac{\mathfrak{v}_\varphi}{r^2 \sin^2 \vartheta}\right) \cdot \tag{35}$$

We can eliminate the factor A/ϱ on the right-hand side by substituting $\frac{A}{\varrho}\mathfrak{v}_r^*$, $\frac{A}{\varrho}\mathfrak{v}_\vartheta^*$, $\frac{A}{\varrho}\mathfrak{v}_\varphi^*$, for $\mathfrak{v}_r$, $\mathfrak{v}_\vartheta$, $\mathfrak{v}_\varphi$. We shall omit the asterisks in the following.

[1] For determination of the distribution of density in the case of uniform rotation and polytropic state, see M. N. 93, 390 (1933).

So the differential equation takes the form

$$F(\mathfrak{v}_\varphi) = \Delta \mathfrak{v}_\varphi - \frac{\mathfrak{v}_\varphi}{r^2 \sin^2 \vartheta} \tag{36}$$

where

$$F(\mathfrak{v}_\varphi) = (\mathfrak{w}, \operatorname{grad} \mathfrak{v}_\varphi) + \left(\mathfrak{v}_r + \frac{\mathfrak{v}_\vartheta}{\operatorname{tg} \vartheta}\right) \frac{\mathfrak{v}_\varphi}{r} \tag{37}$$

and the solution may be written as :

$$\mathfrak{v}_\varphi = \Sigma \Psi_n. \tag{38}$$

If we neglect the left-hand side in first approximation, the equation to be solved takes the form :

$$\Delta \Psi_0 - \frac{\Psi_0}{r^2 \sin^2 \vartheta} = 0 \tag{39}$$

and the solution will be :

$$\Psi_0 = \omega \, r \, P_1^{(1)} = \omega \, r \sin \vartheta$$

where ω (angular velocity) is constant. The second approximation is given by $\Psi_0 + \Psi_2$ and we obtain the following differential equation for Ψ_2 :

$$F(\Psi_0) = \Delta \Psi_2 + \frac{\Psi_2}{r^2 \sin^2 \vartheta}. \tag{40}$$

The n-th approximation will be given by the sum of the first n term of (32) and for ψ_n we have the following differential equation :

$$F(\Psi_{n-1}) = \Delta \Psi_n - \frac{\Psi_n}{r^2 \sin^2 \vartheta}. \tag{41}$$

The solution of the differential equations (40) and (41) is given by the following series

$$\Psi_n = \Sigma \frac{R_{nl}}{r} P_n^{(1)}. \tag{42}$$

The left-hand side of (41) is developed in series of the spherical harmonics after Fourier's principle :

$$F(\Psi_{n-2}) = \Sigma \, Q_{nl} \, P_l^{(1)} \tag{43}$$

where

$$Q_{nl} = \int_0^\pi F(\Psi_{n-2}) \, P_l^{(1)} \sin \vartheta \, d\vartheta \tag{44}$$

and we have the following differential equation for R_{nl} :

$$R_{nl}'' - \frac{l(l+1)}{r^2} R_{nl} = r \, Q_{nl} \tag{45}$$

Q_{nl} leads to very complicated expressions, therefore, we calculate it only in the second approximation. In this approximation :

$$Q_{2l} = \omega_0 \int_0^{\pi} F\,(r \sin \vartheta)\; P_l^{(1)} \sin \vartheta \, d\vartheta \tag{46}$$

(45) may be solved by variation of parameters similarly to (18) :

$$R_{nl} = \frac{r^{l+1}}{2\,l+1} \int_0^r \frac{Q_{nl}}{r^{l-1}} \, dr \quad \text{or} \quad R_{nl} = -\frac{r^{-l}}{2\,l+1} \int_0^r r^{l+2} \, Q_{nl} \, dr \tag{47}$$

Let us insert (39) into (37) :

$$F\,(r \sin \vartheta) = 2\,(\mathfrak{v}_r \sin \vartheta + \mathfrak{v}_\vartheta \cos\vartheta).$$

If we keep only terms of the first order in (34) :

$$\varrho\, \mathfrak{v}_r = 3 \,\frac{b_{20}}{\varkappa_{20}}\, \frac{1}{r^2}\, \varphi_{20}\, (3 \cos^2\vartheta - 1)$$

$$\varrho\, \mathfrak{v}_\vartheta = 3\, \frac{b_{20}}{\varkappa_{20}}\, \frac{1}{r}\, \frac{d\varphi_{20}}{dr}\, \sin\vartheta \, \cos\vartheta$$

Inserting these and using the notation $\varkappa_{20} r = r$, we have

$$F\,(r \sin \vartheta) = 3\, \frac{\omega}{\varrho}\, b_{20} \left[\frac{\varphi_{20}}{r^{12}}\, (3 \cos^2 \vartheta - 1) \sin \vartheta + \frac{3}{r}\, \frac{d}{dr}\, \varphi_{20} \sin \vartheta \cos^2 \vartheta \right]$$

The expansion in series of spherical harmonics is easier if we develop only $\cos^2 \vartheta \sin \vartheta$ and $\sin \vartheta$ in Legendre's associated function of the first order and arrange the expression, so we get :

$$F\,(r \sin \vartheta) = \left[\frac{3}{5}\, \frac{1}{r} \left(\varphi' - 2\, \frac{\varphi}{r}\right) P_1 + \frac{6}{5}\, \frac{1}{r} \left(\varphi' + 3\, \frac{\varphi}{r}\right) P_3^{(1)} \right] \tag{48}$$

respectively :

$$Q_{21} = \frac{3}{5}\, \frac{1}{r} \left(\varphi' - 2\, \frac{\varphi}{r}\right)$$

$$Q_{23} = \frac{6}{5}\, \frac{1}{r} \left(\varphi' + 3\, \frac{\varphi}{r}\right)$$

where :

$$\varphi = \frac{\omega}{\varrho}\, b_{20}\, \varphi_{20}$$

and by (47) if ϱ is constant :

$$R_{21} = \frac{1}{5}\, r\, \varphi - \frac{4}{5}\, \frac{1}{r} \int_0^r r\, \varrho \, dr$$

$$R_{23} = \frac{6}{35}\, r\, \varphi - \frac{6}{35}\, \frac{1}{r^3} \int_0^r r^3\, \varphi \, dr \tag{49}$$

Now from the theory of Bessel's functions:

$$\varphi_{20} = \left(\frac{3}{\varkappa_{20}^2 r^2} - 1\right) \sin \varkappa_{20} r - \frac{3}{\varkappa_{20} r} \cos \varkappa_{20} r$$

Inserting it into (49) after integration we may express the φ component of the velocity in the first approximation:

$$\mathfrak{v}_\varphi = \left(\omega_0 r + \frac{R_{21}}{r}\right) P_1^{(1)} + \frac{R_{23}}{r} P_3^{(1)}$$

Instead of the spherical functions let us introduce the trigonometrical functions and let us arrange it according to powers of cos ϑ:

$$\mathfrak{v}_\varphi = r \sin \vartheta \left[\omega_0 + \frac{1}{r^2}\left(R_{21} - \frac{3}{2} R_{23}\right) + \frac{15}{2} \frac{R_{23}}{r^2} \cos^2 \vartheta\right] \tag{50}$$

hence the two functions w_0 and w_2, we have introduced in order to express the angular velocity, are:

$$w_0 = \omega_0 + \frac{1}{r^2}\left(R_{21} - \frac{3}{2} R_{23}\right)$$

$$w_2 = \frac{15}{2} \frac{R_{23}}{r^2}$$

According to (13) the functions w_0 and w_2 are not sufficient to specify the angular velocity but more coefficients w_n of series (14) of the powers of cos are necessary for this purpose. Coefficients of higher order may be obtained by successive approximations, making use of the aproximate formula (48), we determine first $\mathfrak{c}_\varphi$ and $\mathfrak{A}_\varphi$ in the same way as outlined in §. 2, thereafter using them as in §. 5. we obtain another approximation formula for $\mathfrak{v}_\varphi$. But to carry out this procedure would lead to a very complicated numerical calculation. Therefore we have to be content to develop the approximate form of generalised laws from and formula. (48)

We examine recipropical relations between the constants ($\varkappa_{20}$, b_{20}, ω and A/ϱ) as well as their relations to the other constants characterising the internal structure of the stars. The eigenvalue $\varkappa_{20}$ has been introduced by the eigenfunction φ_{20}. Taking the lower limit of the stellar atmosphere for the star's surface the boundary condition discussed in §. 6. cannot stand, for density on the surface of the star is not zero, it is not even constant, moreover it depends on the direction of the velocity of the large-scale current. Upon the star's surface $\varrho\mathfrak{c}_\varphi$ is not zero and consequently the left-hand side of (31) is in general not equal to zero upon the surface of the star either. Thus it is not possible to develop it by means of orthogonal functions which are equal to zero on the surface. We have to choose the boundary condition so that the function φ_{20} should not be zero upon the surface, but their total sum formed in $\mathfrak{A}_\varphi$ by the corresponding coefficients should be zero, naturally in this case c_φ cannot be zero on the surface. It follows that to determine φ_{20} we have to know the surface value of c_φ and consequently the distribution of the angular velocity and the coefficient b_{20}.

Considering only pure mechanical effects b_{20} and ω_0 are independent constants. As a result of a long calculation b_{20} and ω disappear after integration. It can be assumed without any difficulty that the angular velocity represents a given date of the stellar structure like the radius of the star. But it is far more difficult to explain b_{20} as a constant given by the stellar structure. It is probably necessary to extend this theory further including the electromagnetic forces too. As we know from *Alfvén*'s investigations in case of electromagnetic effects there can be only such differential rotations where the surfaces of constant angular velocity (isorotational surfaces) run along the magnetic lines of force. If this condition can be satisfied within the mechanical limits assumed in this paper, we can reduce the constant b_{20} to the magnetic moment i. e. to the angular momentum, hence, finally to the angular velocity.

A/ϱ depends on the mechanical and thermodynamical stability of the stellar interior and has no relation to the rotation (its change may have). We have to regard it also as a constant given by the stellar structure, the value of which is about 10^{13} CGS units in the case of the sun.

9. The distribution of the angular velocity in the sun's interior. (48) provides a possibility to draw conclusions, from the distribution of the angular velocity on the sun's surface, for the angular velocity in the sun's interior. Disregarding the differential rotation we obtain the radial term of the angular velocity, which increases in the stellar interior. Its function of distribution is shown by the stressed curve of figure Nr. 1. Differential rotation varies also with r and its distribution is shown in figure Nr. 2.

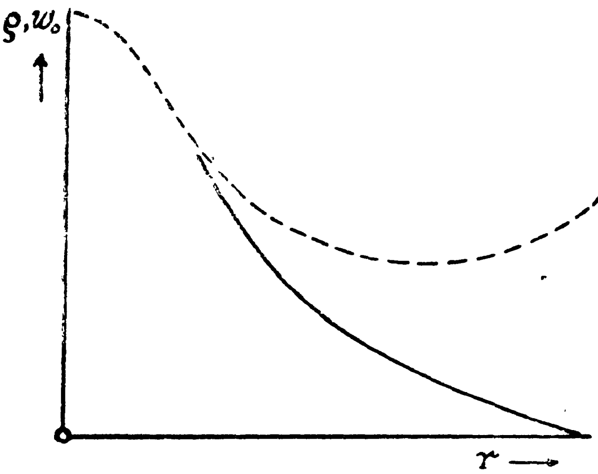

Fig 1,

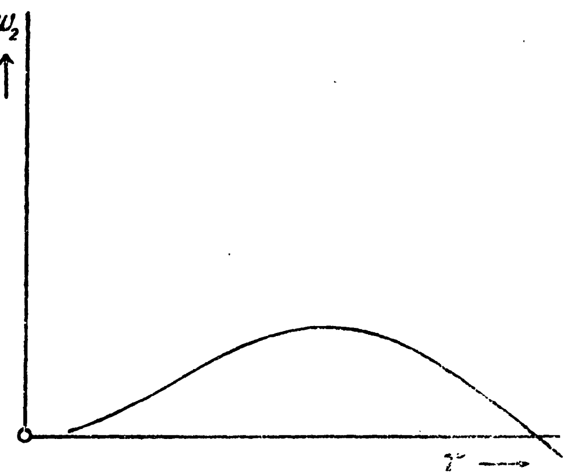

Fig 2

From the equation of energy we may infer that ω and ϱ are approximately proportional. The distribution of density is shown on figure Nr. 1. beside the angular velocity (dotted line) in the case of a non-rotating adiabatical model where $\gamma = 4/3$.

In the case of the sun we may calculate b_{20} if we compare (49) with Faye's interpolation-formula. So we obtain :

$$\frac{15}{2} \frac{\omega \, b_{20}}{\varrho} \frac{R_{23} \, (r_{\odot})}{r} = -0{\cdot}4$$

and

$$r_{\odot} \, \omega_0 + \frac{\omega \quad b_w}{\varrho \, r^2} \left(R_{21} \, (r_{\odot}) - \frac{3}{2} R_{23} \, (r_{\odot}) \right) = 2{\cdot}0$$

from which

$$\frac{b_{20}}{\varrho} = 0{\cdot}1 \;\; r^3$$

We may calculate the angular velocity ($\dot{\vartheta}$) of the meridional current on the surface at $\vartheta = 60°$

$$\dot{\vartheta} = 0{\cdot}2 \, . \, 10^{-10}$$

According to the observations of the sunspots the rate of the meridional and the horizontal angular velocity is :

$$\dot{\vartheta} / \omega = 0{\cdot}38 \, . \, 10^3$$

and its calculated value is :

$$\dot{\vartheta} / \omega = 0{\cdot}01 \, . \, 10^3$$

10. Notes concerning the generation of the sunspots and the solar cycle. If the meridional current of a star has a definite distribution of velocity due to mechanical, thermodynamical and electrodynamical influences, the star can oscillate around this stability state similarly to the pulsation. Its state undergoes a change in consequence of the pulsation ; but there is no change of state through the oscillation of the rotation. As an illustration: the former corresponds to the longitudinal, the latter to the transversal oscillation of a rod. Accordingly the period of the pulsation is short (some days) that of the rotational oscillation is very long (several years).

The difference between the angular velocity of the equatoreal zone and that of the polar areas changes also periodically with the change of the velocity of the meridional current. The atmosphere follows this state of motion with a difference of phase, as it does not take part in dynamical sense in the meridional current (it has no moment- and heat-transports), This means practically that the distribution of velocity has in approximation a discontinuity on the internal boundaries of the atmosphere. Possibilities are offered by this circumstance for the explanation of the generation of sunspots on pure mechanical sense, based on the turbulence theory. On the discontinuity surface (a very high velocity gradient) eddies are generated whose circulation depends statistically on the direction of the velocity gradient. The direction of the velocity gradient changes during a period.

At first, when the inner part is rotating faster, the direction of the gradient shows to the poles, in the second phase when the atmosphere took up already the velocity of the inner part and begins to move more slowly, the velocity gradient shows to the equator. Accordingly the circulation of the eddies changes twice during a period. The eddies are generated on the inner boundary surface of the atmosphere and the convective layer with its up-and-down streamings is situated over them. It is well known that vortex-free circulations may be generated very easily in the case of up-and-down streamings (e. g. water running out of a tube). It is quite probable that the eddies generated on the discontinuity layer arrive to the convective layer by means of the acceleration of these circulations. Therefore, in accordance with the observations sunpots are always generated from the dark, downward streaming particles of the intergranular region.

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