A KONKOLY
CSILLAGVIZSGÁLÓ INTÉZET
KÖZLEMÉNYEI

CONTRIBUTIONS
FROM THE
KONKOLY OBSERVATORY

BUDAPEST-SZABADSÁGHEGY (SVÁBHEGY)

NO. 19

ON THE STATIONARY AND PERIODICAL MOTIONS IN THE ATMOSPHERE OF THE SUN

BY
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1948

A NAP LÉGKÖRÉNEK STACIONÁRIUS ÉS PERIODIKUS MOZGÁSAI.

Jelen dolgozat a Nap légkörének stacionárius és periodikus változásaival foglalkozik a turbulencia-elmélet felhasználásával. A bevezető részben a fotoszféra granulációs szerkezetével és a naptevékenység 11 éves periódusával kapcsolatos termodinamikai és mechanikai instabilitás problémája van összefoglalva. A második rész a turbulens mozgások Taylor-Prandtl-féle alapegyenletét tartalmazza. A harmadik részben a fotoszféra áramlásaira egy modell van kidolgozva, melyből egyrészt a turbulens állapotot jellemző viszkozitási együttható (exchange) a megfigyelések felhasználásával kiszámítható, másrészt a szögsebesség eloszlásának periodikus változására lehet következtetni. E változás számított periódusa nagyságrendben megegyezik a naptevékenység periódusával.

ON THE STATIONARY AND PERIODICAL MOTIONS IN THE ATMOSPHERE OF THE SUN.

By *I. K. Csada.*

Extract. The present paper applies the turbulence-theory to the stationary and periodical motions in the atmosphere of the sun. In the introductory passage the author has summarized the problems of thermodynamical and mechanical instability connected with the 11 years' period of the solar activity and with the granular structure of the photosphere. The second part contains *Taylor-Prandtl*'s equations of the turbulent motions. The third part contains a model relating to large-scale currents in the photosphere; from this the coefficient of exchange (i. e. eddy diffusivity) characteristic of the turbulent state may be calculated on the one hand, and conclusions relating to the periodical changes in the distribution of angular velocity may be drawn on the other hand. The period is approximately equal to the period of solar activity.

1. Introduction. The photosphere of the sun resembles some liquid (or gas) in turbulent state. This state is the result of convective currents taking place in the upmost layers. Such currents occur when there is no thermodynamical equilibrium in the layer. From *Schwarzschild*'s investigations[1] it is well known that a gaseous-sphere of homogeneous structure is always characterized by thermodynamical equilibrium, that is the thermal energy cannot pass from the centre towards the surface but through radiation or, in a lesser degree, through conduction. But, according to *Unsöld*'s investigations, only the structure of the sun's interior can be regarded as homogeneous. In the photosphere and in the closely underlying layers, owing to a mixture of neutral and ionized atoms[2] thermodynamical equilibrium can not be maintained, convective currents arise and thermal energy passes not only by radiation and conduction but also by convection. Both laboratory and theoretical

[1] Gött. Nach. 1916. p. 41.
[2] As proved by Unsöld, Z. f. Aph. 1. 138, 1930.

researches prove that the granular structure of the sun's photosphere is caused by these convective currents[1].

While granular structure is the result of thermodynamical instability, the activity of solar spots and the adjoining phenomena may be considered as results of mechanical instability. According to *Wasiutyński*[2], mechanically unstable regions are being formed as the result of rotation, in the interior of the sun. Gas particles accordingly perform a turbulent motion which is similar to convective motion. We may suppose that, as a result of these two motions, only few granules are abnormally strengthened, which, according to *Rosseland*[3] may lead to the formation of spots. The relations between spots and mechanical (rotationary) processes are also shown by the position of the former. The relations of the phenomena are best shown by *Halm*'s statistical investigations.[4] According to these the distribution of angular velocity of the solar surface, or more accurately, that of the reversing layer, changes with solar activity.

Now we shall try to deduce this change of angular velocity from the periodical solution of the equations of motion. But first we shall give a brief summary of the theory of turbulent motions.

2. Fundamental equations. It is a general characteristic of the turbulent state that neighbouring gas particles of various energy do not only exchange their energies by molecular motion but also by comm n macroscopic motion. Density, pressure, potential and velocity are in steady change and so their mean change may be examined in a macroscopic volume element.

Let $\bar{\varrho}$, $\bar{P}$, $\bar{V}$ and $\bar{\mathfrak{v}}$ be the mean values for density, pressure potential and velocity in the turbulent state and let ϱ', P', V' and $\mathfrak{v}'$ be the changes of pressure, density etc. In that case we may write :

$$\varrho = \bar{\varrho} + \varrho', \qquad P = \bar{P} + P', \qquad V = \bar{V} + V', \qquad \mathfrak{v} = \bar{\mathfrak{v}} + \mathfrak{v}'. \tag{1}$$

Substituting these relations in Euler's equations of motion, by using the definitions

$$\bar{\varrho}' = 0, \quad \overline{P'} = 0, \quad \overline{V'} = 0, \quad \overline{\varrho \mathfrak{v}} = \bar{\varrho}\, \bar{\mathfrak{v}}, \quad \bar{\mathfrak{v}}' \neq 0 \tag{2}$$

we get

$$\bar{\varrho} \frac{\partial \bar{\mathfrak{v}}}{\partial t} + \bar{\varrho}\, \bar{\mathfrak{v}}\, \mathrm{grad}\, \bar{\mathfrak{v}} = \bar{\varrho}\, \mathrm{grad}\, \bar{V} - \mathrm{grad}\, \bar{P} + \overline{[\varrho \mathfrak{v}', \mathrm{rot}\, \mathfrak{v}']} \tag{3}$$

[1] Ap. N. **4.** 135, 1946.
[2] Ap. N. **4.** 91.
[3] Ap. N. **4.** 73.
[4] M. N. **82.** 479, 1929.

where we have neglected ϱ' grad V' as secondary quantity and grad $|\mathfrak{v}'|^2$, the energy of turbulent motion (per mass units).

Taylor suggested that in turbulent state the rotation of velocity (rot $\mathfrak{v}$, vorticity) remains constant along the mixing path. Vorticity transported by collective motion is equal to the mean vorticity at the starting place of the set (rot $\bar{\mathfrak{v}}_1$) and if mean vorticity is rot $\bar{\mathfrak{v}}_2$ at the end of the mixing path, then, in consequence of transportation, it will change to rot $\bar{\mathfrak{v}}_2$. Thus for the change of vorticity of the set we get :

$$\text{rot } \mathfrak{v}' = \text{rot } \bar{\mathfrak{v}}_1 - \text{rot } \bar{\mathfrak{v}}_2.$$

But mean vorticity is an analytical vectorial function of the space, so we can write:

$$\text{rot } \bar{\mathfrak{v}}_2 = \text{rot } \bar{\mathfrak{v}}_1 + \mathfrak{l}\,\mathfrak{grad}\,\text{rot } \bar{\mathfrak{v}}_1.$$

On the basis of this we get the relation of the modified vorticity transport theory

$$\text{rot } \mathfrak{v}' = - \mathfrak{l}\,\mathfrak{grad}\,\text{rot } \bar{\mathfrak{v}} \tag{4}$$

where $\mathfrak{l}$ is the «mixing length», the distance between the beginning and the end of the mixing path.

Substituting (4) in (3) we get a very complicated expression, which, however, becomes simpler and analogous to Stokes-Navier's equation of the viscous liquids if turbulence is isotropic[1]. In that case we get :

$$\frac{d\bar{\mathfrak{v}}}{dt} = \text{grad } \overline{V} - \frac{1}{\varrho} \text{ grad } \overline{P} - \frac{A}{\varrho} \text{ grad div } \bar{\mathfrak{v}} + \frac{A}{\varrho} \Delta \bar{\mathfrak{v}} \tag{5}$$

where A, the coefficient of turbulent viscosity (exchange), is given by

$$A = \varrho\ \overline{|\mathfrak{l}|}\ |\bar{\mathfrak{v}}'|. \tag{6}$$

We can define A/ϱ by the observed and calculated characteristics of the granules. If we substitute the granules by *Bénard*'s convective cell, the mixing length will be the 1/20th part of the mean distance of the granules and $|\mathfrak{v}'|$ may be determined from the difference of temperature (or brightness) between granular and intergranular space, which is about 2·1 km/sec., from which we get

$$\frac{A}{\varrho} = 4{,}3 \cdot 10^{11} \text{ (cgs)}.$$

A/ϱ may be estimated also on the basis of ten *Bruggencate*'s and *Grotrian*'s observations. The mean life-time of the granules is about

[1] The definition of isotropical turbulence see in *Wasiutyński*'s work, Ap. N. 4. 23.

3·2 minutes[1] and the turbulent velocity[2] $|\mathfrak{v}'| = 1{\cdot}8$ km/sec and so

$$|\mathfrak{l}| = \tau \; |\mathfrak{v}'|$$

from which we obtain

$$\frac{A}{\varrho} = \tau \; |\mathfrak{v}'|^2 = 6{,}221 \, . \, 10^{12} \text{ (cgs)}. \tag{7}$$

In the following we shall also determine A/ϱ from the meridional large-scale currents of the sun's atmosphere.

In the equation (5) we have neglected the electromagnetic forces, which originate from the sun's permanent magnetic field on the one hand and the eddy motion of the ions on the other hand. We suppose that in the first approximation the rotation is characterised by the mechanical phenomena.

3. **The motion of the convective layer.** The basis of our investigations will be the vectorial equation (5). Writing this in polar coordinates of space the horizontal component takes the form :

$$\frac{\partial \mathfrak{v}_\varphi}{\partial t} + \mathfrak{v}_r \frac{\partial \mathfrak{v}_\varphi}{\partial r} + \frac{\mathfrak{v}_\vartheta}{r} \frac{\partial \mathfrak{v}_\varphi}{\partial \vartheta} + \frac{\mathfrak{v}_r \, \mathfrak{v}_\varphi}{r} + \frac{\mathfrak{v}_\vartheta \, \mathfrak{v}_\varphi}{r \operatorname{tg} \vartheta} = \frac{A}{\varrho} \left(\Delta \mathfrak{v}_\varphi - \frac{\mathfrak{v}_\varphi}{r^2 \sin^2 \vartheta} \right). \tag{8}$$

The radial and meridional change of $\mathfrak{v}_\varphi$ are very small, thus the terms $\frac{\partial \mathfrak{v}_\varphi}{\partial r}$ and $\frac{\partial \mathfrak{v}_\varphi}{\partial \vartheta}$ may be neglected as of second order. The components of velocity lying in the meridian-plane can be expressed with the help of a vector-potential (if $\varrho = \text{const}$) :

$$\mathfrak{v} = \operatorname{rot} \mathfrak{A} \tag{9}$$

In his paper quoted above *Randers* had proved that large-scale meridional currents can be stable only in stars performing rotation. Unfortunately difficulties in the technique of calculation hinder the quantitative investigations of the distribution of velocity. It may be stated, however, that along the axis of rotation and along the equatoreal plane the radial component of velocity is zero and $\mathfrak{v}_\vartheta \neq 0$. For future purposes we shall introduce an empirical formula for $\mathfrak{A}$ gained from observations. This formula will be required to describe the motion of the solar atmosphere in accordance with the observations and shall fulfill the above boundary conditions. Such a formula is the following (owing to simple properties of symmetry $\mathfrak{A}_r = 0$, $\mathfrak{A}_\vartheta = 0$) :

[1] Z. f. Aph. 12. 323 (1936.)
[2] Z. f. Aph. 18. 316 (1939).

$$\mathfrak{A}_\varphi = \alpha\,(r_\odot^2 - r^2)\,\sin\,2\,\vartheta \tag{10}$$

where a is an arbitrary constant, $r_\odot$ is the sun's radius. For the velocity components we shall have :

$$\begin{aligned} \mathfrak{v}_r &= 2\,\alpha\,\frac{r_\odot^2 - r^2}{r}\,(3\,\cos^2\,\vartheta - 1) \\ \mathfrak{v}_\vartheta &= \alpha\,\frac{3r^2 - r_\odot^2}{r}\,\sin\,2\,\vartheta. \end{aligned} \tag{11}$$

From *Halm*'s investigations we may conclude that the φ-component of velocity changes synchronously with solar activity. If the mean velocity is denoted by $v_0\,(r, \vartheta)$, then we have :

$$\mathfrak{v}_\varphi = v_0\,(r,\,\vartheta) + f\,(r,\,\vartheta)\,e^{i\Omega t} \tag{12}$$

where $2\pi/\Omega$ is the period of the velocity-changes which must be equal to the period of solar activity (11 years).

After substituting (12) and (11) into (8) and neglecting $\mathfrak{v}_\vartheta f$ as of second order, we obtain the following two differential equations for v_0 and f:

$$\left(r_\odot^2\,\frac{2\cos^2\vartheta - 1}{r^2} + 1\right)v_0 = \frac{A}{2\,\alpha\varrho}\left(\Delta v_0 - \frac{v_0}{r^2\,\sin^2\vartheta}\right) \tag{13}$$

$$\Delta f - \left(i\Omega\frac{\varrho}{A} + \frac{1}{r^2\,\sin^2\,\vartheta}\right)f = 0. \tag{14}$$

a) *The distribution of mean angular velocity.* By putting $v_0 = g(r)h(\vartheta)$ into (13), we have

$$\Delta_r g - \left(\lambda^2 + \frac{\lambda^2\,c^2}{r^2}\right)g = 0$$

and

$$\Delta_t h + \left[\lambda^2\,(c^2 + r_\odot^2) - 2\,\lambda^2\,r_\odot^2\,\cos^2\vartheta - \frac{1}{\sin^2\vartheta}\right]h = 0$$

where

$$\lambda^2 = \frac{2\,\alpha\,\varrho}{A}.$$

The first equation may be solved by the help of the series $g = \sum\,a_n\,(\lambda\,r)^n$. By comparing the coefficients of corresponding powers of λr we have

$$\lambda^2\,c^2 = 2$$

and
$$a_{n+2} = \frac{a_n}{(n+2)\,(n+3)-2}$$

from which
$$g = a_0 r \left(1 + \frac{\lambda^2 r^2}{4} + \frac{\lambda^4 r^4}{112} + \ldots \right). \tag{15}$$

After the substitutions $\cos\vartheta = x$ an $h = (1-x^2)^{1/2}\, y$, the second differential equation becomes

$$(1-x^2)\; y'' - 4\, xy' + [2\, \lambda^2 r_\odot^2 x^2 - \lambda^2 r_\odot^2]\; y = 0$$

(making use of the relation $\lambda^2 c^2 = 2$), from which by substituting $y = \sum b_n x^n$ we obtain

$$y = \omega_0 \left(1 - \frac{\lambda^2 r_\odot^2}{2} x^2 + \ldots \right)$$

that is
$$h = \omega_0 \sin\vartheta \left(1 - \frac{\lambda^2 r_\odot^2}{2} \cos^2\vartheta + \ldots \right) \tag{16}$$

The distribution of velocity, according to equations (15) and (16), has the form :

$$v_0 = \omega_0\, r\, \sin\vartheta \left(1 + \frac{\lambda^2 r^2}{4} \right) \left(1 - \frac{\lambda^2 r_\odot^2}{2} \cos^2\vartheta \right) \tag{17}$$

This is in agreement with observations, i. e. the angular velocity increases with the growth of distance from the centre (atmospheric height) and diminishes with the heliographic latitude $90^\circ - \vartheta$.[1] Thus, the vector-potential (10), in the photosphere at least, is in agreement with observations. On the basis of this conformity we are justified to choose an indirect way for the calculation of A/ϱ, by comparing (17) with the motion of the sun's photosphere.

According to Faye's interpolation-formula the distribution of velocity on the solar surface is

$$v_0 = \sin\vartheta\; (2{,}0 - 0{,}4\, \cos^2\vartheta) \qquad \text{km/sec}$$

which compared with (17), gives

$$\lambda^2 = \frac{0.4}{r_\odot^2}.$$

On the other hand, λ^2 is equal to $2a\varrho/A$. To determine ϱ/A we

[1] See *Waldmeier:* Ergebnisse und Probleme der Sonnenforschung, pp. 45.

must further know α. We have approximated large-scale meridional currents with the help of the second equation of (11), from which, substituting $\mathfrak{v}_\vartheta = \dot{\vartheta} r_\odot$, we have

$$2\alpha = \frac{\dot{\vartheta}}{\sin 2\vartheta}$$

$\dot{\vartheta}$, the angular velocity of large-scale meridional currents cannot be observed directly, we can only infer it from the meridional motions of the sunspots. According to *Waldmeier*[1] the maximum rate ot the meridional motions of the sunspots is 1° per rotation periods. But this includes the proper motions of the sunspots themselves. Taking this into consideration the mean meridional motions of the sunspots will be, 0,14° per rotation periods, i. e.

$$\dot{\vartheta} = 1{,}114 \cdot 10^{-9} \qquad \text{radian/sec.}$$

According to the "Schmetterlingsdiagramm" the limit of the lower zone of sunspots varies between 1° and 10°, the upper limit between 8° and 30° heliographic latitudes; from this — taking into consideration the asymmetry of the curve too — the lower limit may be taken 4°, the upper one 12° thus $\vartheta_1 = 78°$, $\vartheta_2 = 86°$. Forming the mean value of $\dot{\vartheta}$:

$$\overline{\dot{\vartheta}} = \frac{2\alpha}{\vartheta_2 - \vartheta_2} \int_{\vartheta_1}^{\vartheta_2} \sin 2\vartheta \, d\vartheta = 0{,}2747 \cdot 2\alpha$$

we obtain

$$2\alpha = 4{,}0553 \cdot 10^{-9} \text{ (cgs)}$$

that is

$$\frac{A}{\varrho} = 10{,}138\, r_\odot^2 \cdot 10^{-9} = 4{,}898 \cdot 10^{13} \qquad \text{(cgs).}$$

The estimates on A/ϱ from the altitudinal changes of the angular velocity are more uncertain. One among the reasons for this is the fact that it is extremely difficult to state the height of the observed layers of the photosphere. Exact measurements of height are only possible in the upper part of the atmosphere — i. e. in the chromosphere — but the conditions in this differ from those in the convective layer. Further, an exact estimate of A/ϱ in the lower layers of the atmosphere is made impossible by the fact that the angular velocity cannot be measured with direct (spectroscopic) methods. The proper motions of the observed particular objects

[1] Op. cit., p. 137.

(spots, faculae, flocculi) makes the observations uncertain. According to *Waldmeier*[1] the expressions for the angular velocity deduced from the observations of flocculi and faculae are:

$$v_0/r_\odot \cos B = 14{,}54° — 2{,}81° \sin^2 B \qquad \text{(faculae, Greenwich)}$$
$$= 14{,}56° — 2{,}98° \sin^2 B \qquad \text{(Ca flocculi, } Fox\text{)}.$$

Substituting $90°—B = \vartheta$ we obtain in the CGS system

$$v_0/r_\odot \sin\vartheta = 2{,}937.10^{-6} — 0{,}5675.10^{-6} \sin^2\vartheta$$
$$= 2{,}951.10^{-6} — 0{,}6019.10^{-6} \sin^2\vartheta.$$

If we take 1000 km $= 10^8$ cm for the difference of height of the flocculi and faculae, the change of angular velocity along the equator will be

$$\frac{d\omega}{dr} = 4{,}0.10^{-17} \qquad \text{(cgs)}$$

extrapolated to the poles we obtain

$$\frac{d\omega}{dr} = — 30{,}37.10^{-17} \qquad \text{(cgs)}.$$

Proceeding from the equator towards the poles, the angular velocity gradient decreases, but on the polar area neither faculae nor flocculi can be observed; however, on the basis of observations of the faculae and flocculi we may assume that the angular velocity gradient changes sign.

According to (17) the angular velocity at the equator and the poles has the values:

$$\omega_{aequ} = \omega_0\left(1 + \frac{\lambda^2 r_\odot^2}{4}\right), \qquad \omega_{pol} = \omega_0\left(1 — \frac{\lambda^2 r_\odot^2}{4}\right)$$

and the angular velocity gradient becomes:

$$\left(\frac{d\omega}{dr}\right)_{aequ} = \omega_0 \frac{\lambda^2 r_\odot}{2}, \qquad \left(\frac{d\omega}{dr}\right)_{pol} = \omega_0 \frac{\lambda^2 r_\odot}{2}\left(1 — \frac{\lambda^2 r_\odot^2}{2}\right)$$

from which we obtain:

$$\lambda^2 = \frac{2}{r_\odot \omega_0}\left(\frac{d\omega}{dr}\right)_{aequ} = \frac{1{,}9175}{r_\odot^2}; \qquad \frac{A}{\varrho} = 0{,}76\ 10^{13} \text{ (equator)}$$

$$\lambda^2 = \frac{1}{r_\odot^2}\left(1 + \sqrt{1 — 4\frac{r_\odot}{\omega_0}\left(\frac{d\omega}{dr}\right)_{pol}}\right) = \frac{2{,}73}{r_\odot^2}; \quad \frac{A}{\varrho} = 0{,}92.10^{13} \text{ (poles)}$$

[1] Op. cit., p. 46.

Finally we shall also examine the chromospheric turbulence. According to *Waldmeier*[1] the angular velocity and the height of some layers (measured from the reversing layer) have the values :

	angular velocity	height
Na D line	13,75° per day	1500 km
He D «	14,40° « «	7500 «
H «	14,77° « «	12000 «
H, K «	14,75° « «	14000 «

from which the mean angular velocity gradient becomes

$$\frac{d\omega}{dr} = 18{,}349.10^{-17}$$

and

$$\frac{A}{\varrho} = 0{,}4055.10^{13}.$$

b) *The periodical changes of the angular velocity.* According to *Halm*'s investigations the distribution of angular velocity changes in the course of the solar cycle. But these changes must satisfy the principle of conservation of angular momentum. In case of a rotation with variable angular velocity this can be satisfied only with a simultaneous change in the velocity of meridional currents. It must be noted, however, that the change of the angular velocity of rotation is also very small, so that the change in meridional currents may be considered as a second order quantity, and hence negligible.

The differential equation of the periodical variation of angular velocity is given by (14). The equation is analogous to the Fourier-equation of heat-conduction and must be solved accordingly, too.[2] Let us write $f = R_l(r) P_l^{(1)}(\vartheta)$ in (14), $P_l^{(1)}$ being the Legendre's associated functions of the first kind and of the l-th order. We then obtain the following differential equation for R_l:

$$\Delta_r R_l - \left(\frac{l\,(l+1)}{r^2} + i\Omega\frac{\varrho}{A}\right) R_l = 0. \tag{18}$$

After the substitution

$$R_l = \frac{a_l}{r} Q_l^{(1)} e^{\lambda r} - \frac{b_0}{r} Q_l^{(2)}\, e^{-\lambda r} \tag{19}$$

[1] Op. cit., p. 51 and p. 170.
[2] See *Joos:* Lehrbuch der theoretischen Physik, p. 450, or *Lamb:* Hydrodynamics, 1906, p. 558.

we get for $Q_l^{(1)}$ and $Q_l^{(2)}$ the following differential equations

$$\frac{d^2Q_l^{(1)}}{dr^2} - 2\lambda \frac{dQ_l^{(1)}}{dr} - \frac{l(l+1)}{r^2} Q_l^{(1)} = 0 \tag{20}$$

$$\frac{d^2Q_l^{(2)}}{dr^2} + 2\lambda \frac{dQ_l^{(2)}}{dr} - \frac{l(l+1)}{r^2} Q_l^{(2)} = 0 \tag{21}$$

where $\lambda^2 = i\Omega \frac{\varrho}{A}$, from which $\lambda = a - i\alpha$, and $\alpha = \sqrt{\frac{\Omega}{2} \frac{\varrho}{A}}$ (22)

(α and λ are not the same as the constants used in (10) and (15)).

With the help of the substitutions

$$Q_l^{(1)} = \sum_\nu \alpha_{l\nu} (\lambda r)^{\nu - l} \text{ and } Q_l^{(2)} = \sum_\nu b^{l\nu} (\lambda r)^{\nu - l}$$

the solutions of (20) and (21) may be expressed in terms of finite polynomes ; let us take $\lambda r = z$, then the solutions of (20) and (21) will be :

$$Q_1^{(1)} = \frac{a_{10}}{z}(1-z) \qquad Q_1^{(2)} = \frac{b_{10}}{z}(1+z)$$

$$Q_2^{(1)} = \frac{a_{20}}{z^2}\left(1-z+\frac{1}{3}z^2\right) \qquad Q_2^{(2)} = \frac{b_{20}}{z^2}\left(1+z+\frac{1}{3}z^2\right)$$

$$Q_3^{(1)} = \frac{a_{30}}{z^3}\left(1-z+\frac{2}{5}z^2-\frac{1}{15}z^3\right) \qquad Q_3^{(2)} = \frac{b_{30}}{z^3}\left(1+z+\frac{2}{5}z^2+\frac{1}{15}z^3\right).$$

Let us split these into real and imaginary parts, introducing the following symbols :

$$Q_l^{(1)} = \mathfrak{R}_l^{(1)} + i\,\mathfrak{J}_l^{(1)}, \qquad Q_l^{(2)} = \mathfrak{R}_l^{(2)} + i\,\mathfrak{J}_l^{(2)}, \qquad z = ar - iar$$

and writing these into (19) ; we have for R_l, split into real and imaginary parts,

$$rR_l = A(r)\cos(\Omega t - h) + i\,A(r)\sin(\Omega t - h') \tag{23}$$

where

$$A^2(r) = B(r)\,[\mathfrak{Cof}(2ar+\gamma) + \cos(2ar+\gamma')$$

$$B^2(r) = 4\,[(\mathfrak{R}_l^{(1)})^2 + (\mathfrak{J}_l^{(1)})^2]\,[(\mathfrak{R}_l^{(2)})^2 + (\mathfrak{J}_l^{(2)})^2]$$

$$\mathfrak{Tg}\,\gamma = \frac{(\mathfrak{R}_l^{(2)})^2 + (\mathfrak{J}_l^{(1)})^2 - (\mathfrak{R}_l^{(2)})^2 - (\mathfrak{J}_l^{(2)})^2}{(\mathfrak{R}_l^{(1)})^2 + (\mathfrak{J}_l^{(1)})^2 + (\mathfrak{R}_l^{(2)})^2 + (\mathfrak{J}_l^{(2)})^2}$$

$$\operatorname{tg}\gamma' = \frac{\mathfrak{R}_l^{(2)}\mathfrak{J}_l^{(1)} - \mathfrak{R}_l^{(1)}\mathfrak{J}_l^{(2)}}{\mathfrak{R}_l^{(1)}\mathfrak{R}_l^{(2)} + \mathfrak{J}_l^{(1)}\mathfrak{J}_l^{(2)}}.$$

For h and h' we find similarly complicated expressions.

The periodical changes of the convective layer are given by the real or the pure imaginary part of (23):

$$\Re e\, f(r, \vartheta) e^{i\,\Omega\,t} = \frac{A(r)}{r} \cos\, (\Omega t + h)\, P_l^{(1)}.$$

$A(r)$ and h contain three parameters, a_{lo}, b_{lo} and α; to determine these we must know the phenomena taking place on the inner boundary surface of the convective layer. But to determine the period, it is sufficient to know α. (It is easy to understand however that α is strongly dependent on the boundary conditions imposed on (23.) These boundary conditions may be determined from phenomena taking place on the inner boundary surface of convective layer, such as change of velocity, etc.) In the following we shall assume that the amplitude of the change is constant within the whole convective layer.

4. **Discussion.** Let us examine (12) in the case of $l = 1$ and $l = 3$. (The periodical changes must be symmetrical to the equator, and this is only fulfilled if $l \neq 2,4, \ldots.$)

a) $l = 1$, $P_1^{(1)} = \sin\vartheta$. The distribution of velocity by (12) (17) and (23) is.[1]

$$\mathfrak{v}_\varphi = v_0\,(r, \vartheta) + r\, \sin\vartheta\, \frac{A(r)}{r^2} \cos\, (\Omega t + h)$$

where

$$A^2(r) = B(r)\ [\mathfrak{Cof}(2\alpha r + \gamma) + \cos\,(2\alpha r + \gamma')]$$

$$B^2(r) = \left(\frac{1}{4\alpha r}\right)^2 + \frac{1}{4}, \qquad \mathfrak{Tg}\,\gamma = \frac{2\alpha r}{1 + 2\alpha^2 r^2},\ \operatorname{tg}\gamma' = \frac{2\alpha r}{1 - 2\alpha^2 r^2}.$$

In this case the change of angular velocity depends only on $\sin\vartheta$, which means that it is zero at the poles and reaches its maximum value at the equator. According to *Halm*'s investigations the distribution is the very contrary, a decision whether this distribution is physically possible, or not, may be only reached by the examination of the principle of conservation of angular momentum. Some difficulties seem to arise also in determining α. The boundary condition imposed on $f(r, \vartheta)$ cannot be satisfied but in the case where $\alpha = 0$. We imposed this boundary condition on the principle that the change should be the smallest possible.

b) $l = 3$. The change of velocity-distribution takes the form:

$$\omega_0 r_\odot \sin\vartheta\, [1 + \tfrac{3}{2}\, c\, \cos\,(\Omega t + h) - (\varkappa + \tfrac{15}{2}\, c) \cos\,(\Omega t + h) \cos^2\vartheta].$$

where
$$\varkappa = \frac{b}{a} \quad \text{and} \quad c = A\,(r_\odot)/r_\odot^2.$$

At the poles the change is five times greater than at the equator. Functions introduced for the calculations of $A(r)$ and the other functions are:

$$\mathfrak{R}_3^{(1)} = \frac{a_{30}}{x^3}\left(\frac{1}{4} - \frac{1}{5}x^2 - \frac{1}{15}x^3\right) \qquad \mathfrak{J}_3^{(1)} = -\frac{a_{30}}{x^2}\left(\frac{1}{4} - \frac{1}{2}x - \frac{1}{5}x^2\right).$$

$$\mathfrak{R}_3^{(2)} = -\frac{b_{30}}{x^3}\left(\frac{1}{4} - \frac{1}{5}x^2 - \frac{1}{15}x^3\right) \qquad \mathfrak{J}_3^{(2)} = -\frac{b_{30}}{x^3}\left(\frac{1}{4} + \frac{1}{2}x + \frac{1}{5}x^2\right)$$

From this the zero-value of the differential-quotient of $R(r)$ is $2ar = 0{,}9366$. The a must be chosen in such a manner that the velocity-oscillation on the inner boundary surface of the convective layer should be zero.

5. **Comparison with observations.** According to spectroscopic observations during the activity cycle 1901—1913, the values of the coefficients a and b in Faye's interpolation formula for the angular velocity of the reversing layer are tabulated below:

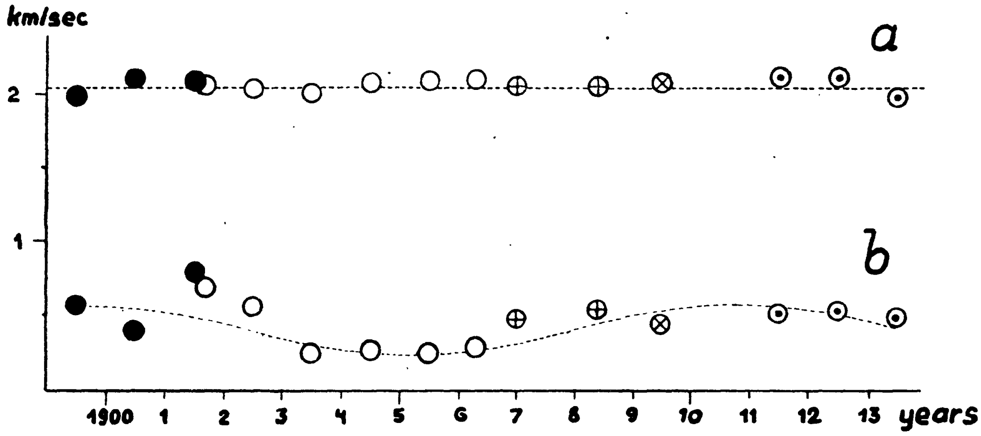
year	a	b	Literature
1899,5	1,98	0,57	A. N. **173**, 272
1900,5	2,11	0,40	«
1901,5	2,09	0,79	«
1901,7	2,06	0,70	«
1092,5	1,979	0,560	«
1903,5	2,036	0,251	«
1904,5	2,075	0,271	«
1905,5	2,093	0,245	«
1906,3	2,010	0,294	«
1907,0	2,055	0,48	Ap. J. **42**, 373
1908,4	2,05	0,55	«
1909,5	2,08	0,45	«
1911,5	2,012	0,528	«
1912,5	2,012	0,541	«
1913,5	1,988	0,489	«

Taking a, the angular velocity on the equator, constant and evaluating b graphically from fig. 1. we get for the change of the velocity the formula:

$$\mathfrak{v}_\varphi = \sin\vartheta\ (2{,}040 - (0{,}41 + 0{,}16 \cos(\Omega t + \beta)) \cos^2\vartheta).$$

In later spectroscopic observations the question of the connection between solar activity and angular velocity as depending upon latitude has been, unfortunately, omitted. The observations of 1914 were only concentrated on the equatorial area.

It may be noted, that according to (25) the angular velocity along the equator changes periodically and with a very small amplitude (five times smaller than at the poles). It was *Halm* who in his work mentioned above inferred such effects. To make clear this problem the next two cycles had been observed at Mount Wilson and in Edinburg.[1] These series of observations of completely homogeneous character, but extended only over the equator, show no effect of this kind, not even statistically. Notwithstanding we have no reason to doubt the correctness of the theory because of this contradiction. The reason of the difference may be sought in the fact that A/ϱ is also dependent on ϑ, because part of the latter originates from mechanical (rotationary) processes.



6. **The period of the change of angular velocity and the solar cycle.** According to (22) the period of the change of angular velocity is

$$\Omega = 2\alpha^2 \frac{A}{\varrho}.$$

As a result of the boundary condition imposed under 4. we get

$$\alpha r_1 = 0{,}9366$$

where r_1 is the distance of the convective layer from the solar centre. But the depth of the convective layer is not more than 700 km, 1/1000 part of the sun's radius, thus instead of r_1 we may take the radius of the sun $r_\odot = 6{,}951 \cdot 10^{-10}$ cm and so

$$\alpha = 1{,}3474 \cdot 10^{-11} \qquad \text{(cgs)}.$$

Values deduced for A/ϱ in 2. and 3.:

[1] *J. Storey*, M. N. **92.** 737.

1. $A/\varrho = 0{,}62 \cdot 10^{13}$ from the observations of granules (P. ten Bruggencate and W. Grotrian)

2. $A/\varrho = 0{,}76 \cdot 10^{13}$ from the change of the angular velocity with atmospherical height in the equator.

3. $A/\varrho = 3{,}63 \cdot 10^{13}$ taking mixing length equal to the mean distance of granules

4. $A/\varrho = 4{,}898 \cdot 10^{13}$ from large-scale meridional currents.

With the help of these the period of the change of angular velocity will be $(T = 2\pi/\Omega)$:

1. $T = 92{,}80$ years
2. $T = 75{,}70$ «
3. $T = 15{,}85$ «
4. $T = 11{,}74$ «

The period of solar activity is 11,1 years which corresponds fairly well to the period received from the exchange value calculated indirectly from large-scale meridional currents.

In addition to the above considerations we shall examine how the uncertainty of the value of a influences the period. If the zero-value of the amplitude is not on the inner boundary surface of the convective layer but in the deeper strata of the sun, the corresponding values of the period and of a will be the ones given in the following table (d, the zero-place of the amplitude is at a distance of the 1/1000, 1/100, 1/10, 2/10, 4/10 and 6/10th part of the sun's radius, from the surface):

$d/r_\odot$	1/1000	1/100	1/10	2/10	3/10	4/10	
a	1,3488	1,3610	1,4971	1,6843	2,2457	4,3685	
1.	88,26	86,67	71,63	56,60	31,84	14,15	years
2.	72,00	70,70	58,44	56,17	25,97	11,54	«
3.	15,08	14,80	12,24	9,67			«
4.	11,17	10,98	9,07				«

It is with sincere pleasure that I express my gratitude to Professor Svein *Rosseland*, who has read the manuscript, contributing to it invaluable criticism.

Konkoly Observatory,
Budapest, 1948 May.

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